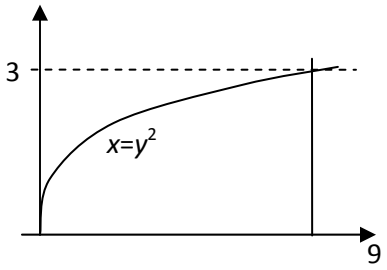
1. **(14pts)** Sketch the region of integration, change the order of integration, and evaluate the integral:

$$\int_0^3\int_{y^2}^9 y\cos\left(\frac{\pi}{2}x^2\right)dx\,dy=\int_0^9\int_0^{\sqrt{x}} y\cos\left(\frac{\pi}{2}x^2\right)dy\,dx=\frac{1}{2}\int_0^9 x\cos\left(\frac{\pi}{2}x^2\right)dx=\frac{1}{2\pi}\sin\left(\frac{\pi}{2}x^2\right)\Big|_0^9=\frac{1}{2\pi}\sin\left(\frac{81\pi}{2}\right)=\boxed{\frac{1}{2\pi}}$$



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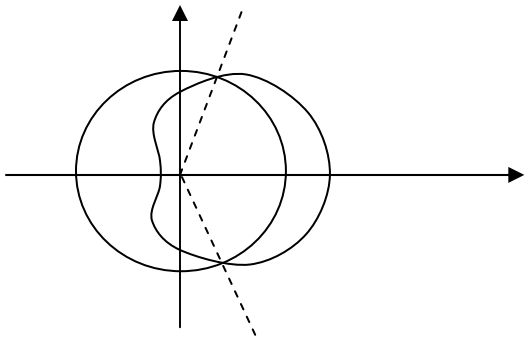
2. **(14pts)** Sketch and use double integration in polar coordinates to find the area enclosed inside the curve $r=3+2\cos\theta$ and outside the circle r=4.

The Limacon $r=3+2\cos\theta$ and the circle $r=4$ intersect when $2\cos\theta=1 \;\Rightarrow\; \theta=\pm\frac{\pi}{3}$

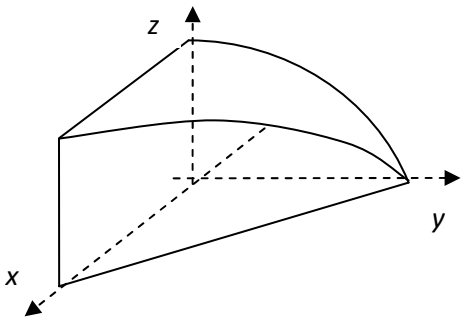
$$\int_{-\pi/3}^{\pi/3}\int_4^{3+2\cos\theta} r\,dr\,d\theta=2\int_0^{\pi/3}\int_4^{3+2\cos\theta} r\,dr\,d\theta=\int_0^{\pi/3} r^2\Big|_4^{3+2\cos\theta}\,dr=\int_0^{\pi/3}\left[(3+2\cos\theta)^2-16\right]dr=$$

$$=\int_0^{\pi/3}\left[-7+12\cos\theta+4\underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}\right]dr=\int_0^{\pi/3}\left[-5+12\cos\theta+2\cos 2\theta\right]dr=\left[-5\theta+12\sin\theta+\sin 2\theta\right]_0^{\pi/3}$$

$$=-\frac{5\pi}{3}+6\sqrt{3}+\frac{\sqrt{3}}{2}=\boxed{\frac{13\sqrt{3}}{2}-\frac{5\pi}{3}}$$



3. **(14pts)** Set up the limits on the triple integral over the region enclosed in the first octant by the surfaces $x + y = 4$ and $y^2 + 4z^2 = 16$, as shown in the figure, in any three different integration orders. Which orders of integration are not possible without breaking the integral into several pieces? (recall that there is a total of six possible integration orders for a triple integral)



$$\int_0^2\int_0^{\sqrt{16-4z^2}}\int_0^{4-y} dx\,dy\,dz = \int_0^4\int_0^{\sqrt{4-y^2/4}}\int_0^{4-y} dx\,dz\,dy = \int_0^4\int_0^{4-y}\int_0^{\sqrt{4-y^2/4}} dz\,dx\,dy = \int_0^4\int_0^{4-x}\int_0^{\sqrt{4-y^2/4}} dz\,dy\,dx$$

The following orders are not possible without breaking up the integration domain: dy dx dz, dy dz dx

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4. Find the mass of an object with volume density $\delta(x, y, x) = xz$ enclosed **in the first octant** inside the cylinder $x^2 + y^2 = 4$ and below the cone $z^2 = x^2 + y^2$, using triple integration in two different coordinate systems: (a) cylindrical (b) spherical.

(a) (14pts) Cylindrical:

$$\int_0^{\pi/2}\int_0^2\int_0^r xz\,dz\,r\,dr\,d\theta = \int_0^{\pi/2}\int_0^2\int_0^r z\,dz\;r^2\,dr\;\cos\theta\,d\theta = \underbrace{\int_0^{\pi/2}\cos\theta\,d\theta}_{1}\int_0^2 \frac{z^2}{2}\Big|_0^r r^2\,dr = \frac{1}{2}\int_0^2 r^4\,dr = \frac{r^5}{10}\Big|_0^2 = \boxed{\frac{16}{5}}$$

(b) (6pts) Spherical:

$$\int_0^{\pi/2}\int_{\pi/4}^{\pi/2}\int_0^{2/\sin\phi} xz\;\rho^2\sin\phi\,d\rho\,d\phi d\theta = \int_0^{\pi/2}\int_{\pi/4}^{\pi/2}\int_0^{2/\sin\phi} \rho^4\sin^2\phi\cos\phi\cos\theta\,d\rho\,d\phi d\theta$$

$$= \underbrace{\int_0^{\pi/2}\cos\theta\,d\theta}_{1}\int_{\pi/4}^{\pi/2}\frac{\rho^5}{5}\Big|_0^{2/\sin\phi}\sin^2\phi\cos\phi\,d\phi = \frac{2^5}{5}\int_{\pi/4}^{\pi/2}\frac{1}{\sin^3\phi}\underbrace{\cos\phi\,d\phi}_{d\sin\phi} = -\frac{2^4}{5}\underbrace{\frac{1}{\sin^2\phi}\Big|_{\pi/4}^{\pi/2}}_{1-2} = \boxed{\frac{16}{5}}$$

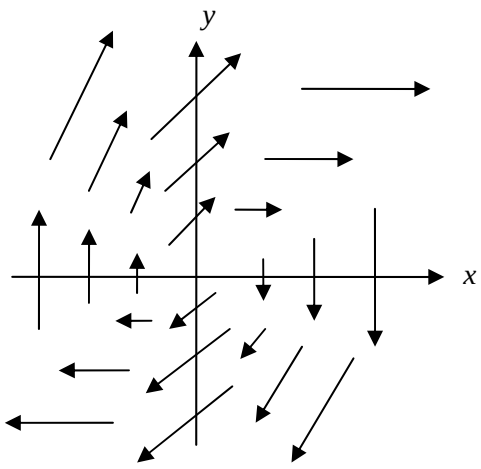
5. **(14pts)** Use coordinate transformation *u*=2*x*–*y*, *v*=*y* to calculate the following integral; make sure to find the region of integration in the *u*-*v* plane

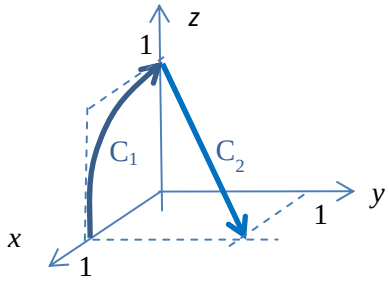
$$\left.\begin{aligned} u &= 2x-y \\ v &= y \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x &= \frac{u+v}{2} \\ y &= v \end{aligned}\right\} \Rightarrow J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$$

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3 (2x-y) e^{(2x-y)^2}\, dx\, dy = \int_0^2 \int_0^4 v^3 u\, e^{u^2} \,|J|\, du\, dv = \frac{1}{2}\int_0^2 \int_0^4 v^3 u\, e^{u^2}\, du\, dv$$

$$= \frac{1}{2}\int_0^2 v^3 dv \int_0^4 u\, e^{u^2}\, du = \frac{1}{4}\frac{v^4}{4}\Big|_0^2 e^{u^2}\Big|_0^4 = \boxed{e^{16}-1}$$

6. **(16pts)** Sketch the vector field $\vec{F}(\vec{r}) = \langle y,\ y-x, 0\rangle$ in the *x*-*y* plane, and calculate its curl and its divergence. Then, find work performed by this force over a piece-wise smooth curve consisting of a quarter-circle C_1 and a straight line C_2, as indicated in the figure:





$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y & y-x & 0 \end{vmatrix} = \langle 0,0,-2\rangle = \boxed{-2\mathbf{k}} \text{ (clockwise)}$$

$$\nabla \cdot \mathbf{F} = 0+1 = \boxed{1}$$

$$C_1:\ \mathbf{r}(t) = \langle \cos t,\ 0,\ \sin t\rangle \Rightarrow \left.\begin{aligned} d\mathbf{r} &= \langle -\sin t,\ 0,\ \cos t\rangle dt \\ \mathbf{F}(\mathbf{r}) &= \langle 0,\ -\cos t,\ 0\rangle \end{aligned}\right\} \Rightarrow \mathbf{F}\cdot d\mathbf{r} = 0$$

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \boxed{0}$$

$$C_2:\ \mathbf{r}(t) = \langle 0,\ 0,\ 1\rangle + t\big(\langle 1,1,0\rangle - \langle 0,0,1\rangle\big) = \langle 0,0,1\rangle + t\langle 1,1,-1\rangle = \langle t,\ t,\ 1-t\rangle$$

$$\Rightarrow \left.\begin{aligned} d\mathbf{r} &= \langle 1,\ 1,\ -1\rangle dt \\ \mathbf{F}(\mathbf{r}) &= \langle t,\ 0,\ 0\rangle \end{aligned}\right\} \Rightarrow \mathbf{F}\cdot d\mathbf{r} = t\ dt$$

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 t\, dt = \frac{t^2}{2}\Big|_0^1 = \boxed{\frac{1}{2}}$$

$$\Rightarrow \int_{C_1\cup C_2} \mathbf{F}\cdot d\mathbf{r} = 0 + \frac{1}{2} = \boxed{\frac{1}{2}}$$

7. **(14 pts)** Consider vector field $\vec{\mathbf{F}}(x,y,z)=\left\langle y\sin(xy),\ \ 2(z+1)^{3/2}+x\sin(xy)+z^3,\ \ 3y\sqrt{z+1}+3z^2y\right\rangle$.

a) Find the potential function of this vector field.

b) Find work done by this force over a linear trajectory from point (1, 0, 1) to point (0, π, 0)

$$\left.\begin{aligned}
\frac{\partial f}{\partial x} &= y\sin(xy), && \Rightarrow\ f=-\cos(xy)+h(y,z)\\
\frac{\partial f}{\partial y} &= 2(z+1)^{3/2}+x\sin(xy)+z^3 && \Rightarrow\ f=2y(z+1)^{3/2}-\cos(xy)+yz^3+p(x,z)\\
\frac{\partial f}{\partial z} &= 3y\sqrt{z+1}+3z^2y && \Rightarrow\ f=2y(z+1)^{3/2}+yz^3+q(x,y)
\end{aligned}\right\}\ f=2y(z+1)^{3/2}-\cos(xy)+yz^3$$

$$W=f(0,1,3)-f(1,0,1)=2(3+1)^{3/2}-\cos(0)+3^3-\left(-\cos(0)\right)=2\cdot 8+27=\boxed{43}$$