Read each problem carefully. Show all work for each problem. Electronic devices and notes *not* allowed.

1. **(12pts)** Consider an object moving with acceleration $\boldsymbol{a}(t) = \langle 2,\ 4\sin(2t)\,,\ 6t\rangle$, with initial velocity and position given by $\boldsymbol{v}(0) = \langle 0,\ 0,\ 0\rangle$ and $\boldsymbol{r}(0) = \langle 1,\ 0,\ 2\rangle$.

a) **(6pts)** Find the velocity vector, $\boldsymbol{v}(t)$:

$$\boldsymbol{v}(t) = \int \boldsymbol{a}(t)dt = \langle \mathbf{2t}, \mathbf{-2\cos(2t)}, \mathbf{3t^2}\rangle + \vec{c}$$

$$\boldsymbol{v}(0) = \langle 0,-2,0\rangle + \vec{c} = \vec{0} \ \Rightarrow\ \ \vec{c} = \langle 0,2,0\rangle \ \Rightarrow\ \ \boldsymbol{v}(t) = \langle \mathbf{2t}, \mathbf{2 - 2\cos(2t)}, \mathbf{3t^2}\rangle$$

b) **(6pts)** Find the position vector, $\boldsymbol{r}(t)$.

$$\boldsymbol{r}(t) = \int \boldsymbol{v}(t)dt = \langle \mathbf{t^2}, \mathbf{2t - \sin(2t)}, \mathbf{t^3}\rangle + \vec{C}$$

$$\boldsymbol{r}(0) = \langle 0,0,0\rangle + \vec{C} = \langle 1,0,2\rangle \ \Rightarrow\ \ \vec{C} = \langle 1,0,2\rangle \ \Rightarrow\ \ \boldsymbol{r}(t) = \langle \mathbf{1 + t^2}, \mathbf{2t - \sin(2t)}, \mathbf{2 + t^3}\rangle$$

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2. **(13pts)** For the function $f(x,y) = \ln(4 - x^2 - 4y^2)$

a) **(5pts)** Find and sketch the domain as a region in the (*x*, *y*) plane. Is it an open or a closed region?

The logarithm requires positive values of argument, therefore:

$$4 - x^2 - 4y^2 > 0 \ \Rightarrow\ \ x^2 + 4y^2 < 4 \ \Rightarrow\ \ \frac{x^2}{4} + y^2 < 1$$

It's the interior of an ellipse with half-axes of 2 and 1. The boundary is excluded, so the domain is open.

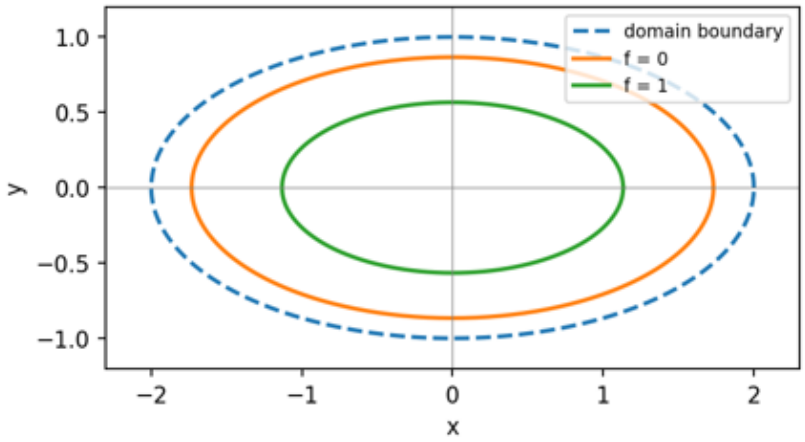
b) **(3pts)** Determine the range of $f(x,y)$.

Logarithm is a monotonically increasing function, therefore the range is determined by the logarithm of the largest and smallest values of its argument within the domain:

$$0 < 4 - x^2 - 4y^2 \le 4 \ \Rightarrow\ -\infty < \ln(4 - x^2 - 4y^2) \le \ln 4 \ \Rightarrow\ \ Range = (-\infty, \ln 4]$$

c) **(5pts)** Find the equations of the level curves corresponding to $f = 0$ and $f = 1$. Make a *rough* sketch of these level curves on the same coordinate axes.

- $f(x,y) = 0$: $4 - x^2 - 4y^2 = e^0 = 1 \ \Rightarrow\ x^2 + 4y^2 = 3$ *(ellipse inside the domain)*
- $f(x,y) = 1$: $4 - x^2 - 4y^2 = e^1 = e \ \Rightarrow\ x^2 + 4y^2 = 4 - e$ *(a smaller ellipse)*



3. **(12pts)** Consider the following two limits. If the limit exists, then find its value. If it does not, then prove this using a two-path test:

(a) $\lim\limits_{(x,y)\to(1,2)} \dfrac{x^2+xy-3x}{x+y-3} = \lim\limits_{(x,y)\to(1,2)} \dfrac{x(x+y-3)}{x+y-3} = \lim\limits_{(x,y)\to(1,2)} x = \mathbf{1}$ (exists)

(b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+xy+y^2}$ does **not** exist: different values along different paths to $(0,0)$:

- Horizontal path, $y=0,\ x\to 0$: $\dfrac{xy}{x^2+xy+y^2} = \dfrac{x\cdot 0}{x^2+x\cdot 0+0^2} = 0$
- Diagonal path, $y=x\to 0$: $\dfrac{xy}{x^2+xy+y^2} = \dfrac{x^2}{x^2+x^2+x^2} = \dfrac{1}{3}$

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4. **(12pts)** Consider the surface $2x^2+3y^2+2z^2+2xz=9$.

a) **(6pts)** Find the tangent plane to this surface at point $P(1,1,1)$.

$$\nabla F = \langle 4x+2z, 6y, 4z+2x\rangle, \quad \vec{n} = \nabla F(1,1,1) = \langle 6,6,6\rangle$$

$$\vec{n}\cdot(\vec{r}-\vec{r_0}) = 6(x-1)+6(y-1)+6(z-1)=0, \qquad \text{which simplifies to } x+y+z=3$$

b) **(6pts)** Use linearization to estimate z on this surface near $P(1,1,1)$ when $x=1.02, y=0.99$.

Tangent plane in fact represents the linearization of $z(x,y)$ near $P(1,1,1)$, therefore:

$$z = 3-x-y = 3-1.02-0.99 = 0.99$$

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5. **(12pts)** Consider the function $f(x,y,z)=xe^y+yz^2$ and points $P(2,0,1)$ and $Q(4,-1,3)$.

a) **(4pts)** Find the gradient of function $f(x,y,z)$ at point P.

$$\nabla f = \langle e^y, xe^y+z^2, 2yz\rangle \ \therefore\ \nabla f(2,0,1) = \langle 1,3,0\rangle$$

b) **(4pts)** Find the maximum rate of change of $f(x,y,z)$ at P, and the unit vector in the direction of this maximum rate of change.

- Maximal rate of change: $||\nabla f(2,0,1)|| = ||\langle 1,3,0\rangle|| = \sqrt{1+9} = \sqrt{10}$
- Unit vector in the direction of maximal rate of change: $\hat{u} = \dfrac{\nabla f(2,0,1)}{||\nabla f(2,0,1)||} = \dfrac{\langle 1,3,0\rangle}{\sqrt{10}}$

c) **(4pts)** Find the directional derivative of $f(x,y,z)$ in the direction from point P toward point Q.

$$\hat{u} = \widehat{PQ} = \frac{\langle 2,-1,2\rangle}{\sqrt{4+1+4}} = \frac{\langle 2,-1,2\rangle}{3}$$

$$D_u f(2,0,1) = \nabla f(2,0,1)\cdot\hat{u} = \frac{\langle 1,3,0\rangle\cdot\langle 2,-1,2\rangle}{3} = \frac{2-3}{3} = -\frac{1}{3}$$

6. **(13pts)** Find all critical points of $f(x,y) = x^3 + y^3 - 3xy$, and classify each as a local maximum, local minimum, or saddle point using the second derivative test.

First, let's find the critical points:

- $f_x = 3x^2 - 3y = 0 \Rightarrow y = x^2$
- $f_y = 3y^2 - 3x = 0 \Rightarrow x = y^2 = (x^2)^2 = x^4 \Rightarrow x = 0 \text{ or } x = 1$

Thus, the critical points are (0, 0) and (1, 1).

Now, apply the second derivative test, noting that $f_{xx} = 6x, f_{yy} = 6y, f_{xy} = f_{yx} = -3$:

Point:	$f_{xx} = 6x$	$f_{yy} = 6y$	$D = f_{xx}f_{yy} - 9$	Classify:
(0, 0)	0	0	$-9 < 0$	Saddle
(1, 1)	$6 > 0$	$6 > 0$	$36 - 9 > 0$	Local min

7. **(13pts)** Use Lagrange multipliers to find the maximum and minimum *values* of $f = 2x - y + 2z$ on the surface of the sphere $x^2 + y^2 + z^2 = 9$. State the points at which such maximum and minimum values are achieved.

$$\nabla f = \lambda \nabla g \Rightarrow \begin{pmatrix} 2 = 2\lambda x \\ -1 = 2\lambda y \\ 2 = 2\lambda z \end{pmatrix} \Rightarrow \begin{pmatrix} x = \frac{1}{\lambda} \\ y = -\frac{1}{2\lambda} \\ z = \frac{1}{\lambda} \end{pmatrix} \Rightarrow x^2 + y^2 + z^2 = \frac{1 + \frac{1}{4} + 1}{\lambda^2} = \frac{9}{4\lambda^2} = 9 \Rightarrow \lambda = \pm\frac{1}{2}$$

Thus, we obtain 2 critical points of $f(x,y,z)$ on the sphere $g(x,y,z) = 9$:

$$\lambda = \frac{1}{2}: \ (x,y,z) = \left(\frac{1}{\lambda}, -\frac{1}{2\lambda}, \frac{1}{\lambda}\right) = (2, -1, 2) \Rightarrow f(2,-1,2) = 2x - y + 2z = 9 \Rightarrow \text{max at } (2,-1,2)$$

$$\lambda = -\frac{1}{2}: \ (x,y,z) = \left(\frac{1}{\lambda}, -\frac{1}{2\lambda}, \frac{1}{\lambda}\right) = (-2, 1, -2) \Rightarrow f(-2,1,-2) = 2x - y + 2z = -9 \Rightarrow \text{min at } (-2,1,-2)$$

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8. **(13pts)** Find the second-degree Taylor polynomial for $f(x,y) = e^x \cos y$ centered at (0,0). Use your polynomial to approximate $f(0.1, 0.2)$.

$$f_x = e^x \cos y, \quad f_y = -e^x \sin y, \quad f_{xx} = e^x \cos y, \quad f_{xy} = -e^x \sin y, \quad f_{yy} = -e^x \cos y$$

$$f(x,y) \approx f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}\left(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right)$$

$$= 1 + x + 0 + \frac{1}{2}\left(x^2 + 0 - y^2\right) = 1 + x + \frac{x^2 - y^2}{2} \Rightarrow f(0.1, 0.2) \approx 1.1 + \frac{0.01 - 0.04}{2} = 1.1 - 0.015 = 1.085$$

$$\text{Let's double-check:} \quad f(x,y) = e^x \cos y \approx \left(1 + x + \frac{x^2}{2} + \cdots\right)\left(1 - \frac{y^2}{2} + \ldots\right) = 1 + x + \frac{x^2 - y^2}{2} + \cdots$$