Math 340 • Fall 2025 • Midterm Exam • October 23, 2025 • Prof. Matveev

Read each problem assignment carefully. Please explain all steps in your solution, briefly.

1) (14pts) Find the first **two non-zero terms** in the Taylor expansion of $\boldsymbol{f(x) = \frac{2x}{\sin(2x)}}$ near $x = 0$, using composition of Taylor expansions of two common elementary functions. Then, use your answer to estimate $\boldsymbol{f(0.6)}$.

2) (14pts) Which of the functions below suffer catastrophic **loss of all significant digits** when computed in double precision for small $|x|$? Write down the precise **condition on** $|\boldsymbol{x}|$ when this happens. Finally, **modify** each expression suffering from this problem so that it provides a more accurate result when evaluated for very small $|x|$:

$$\textit{(a)}\ \ f(x) = \frac{1}{16} - \frac{1}{16 + x^8}\ , \quad \textit{(b)}\ \ g(x) = 1 - \cos(2x^4), \quad \textit{(c)}\ \ h(x) = \frac{1}{1 + \sin(x)}.$$

3) (12pts) Convert $\boldsymbol{x = 37.1}$ from decimal to **binary** number system, keeping 8 digits after the fractional point. Then, write this binary number is standard scientific notation (significant part times an integer power of 2).

4) (14pts) Consider the iteration $\boldsymbol{x_{n+1} = g(x_n)}$ with $\boldsymbol{g(x) = x + \frac{7}{16}(x^2 - 1)}$

a) Find all fixed points of this iteration, and determine their stability (**hint:** fixed points are integers)

b) For each **stable** fixed point, find the **number of iteration** needed to reach the fixed point to within an absolute distance of 2^{-52} (machine precision), assuming initial distance from the fixed point of $\boldsymbol{\epsilon_0 \approx 1/2}$.

Note: logarithms are **not** needed to answer this question.

5) (14pts) Write down a function $f(x)$ whose root (zero) equals $r = 1/\sqrt[3]{4}$. Your function should only involve multiplication, integer powers, and/or division (no fractional powers). Write down the Newton's iteration for finding the root of this function, and perform two iterations to approximate $r = 1/\sqrt[3]{4}$, starting with $\boldsymbol{x_0 = 1}$. Compare your result with the actual value of the root: $1/\sqrt[3]{4} \approx 0.63$.

6) (12pts) Consider 3 different fixed-point iterations methods, each characterized by distinct error behavior, as follows:

$$(a)\ \ \epsilon_{n+1} = 0.01\epsilon_n, \quad (b)\ \ \epsilon_{n+1} = 10\epsilon_n^3, \quad (c)\ \ \epsilon_{n+1} = \epsilon_n^2$$

Compute $\boldsymbol{\epsilon_2}$ for each case, assuming that $\boldsymbol{\epsilon_0 = 10^{-2}}$. How many iterations are needed to reduce the error to the level of $\boldsymbol{\epsilon_n \le 10^{-16}}$, in each of the three cases? Which of the three iterations converges the fastest?

Note: logarithms are **not** needed to answer this question.

7) (14pts) Perform least-squares fit of the model $\boldsymbol{y_{model}(t; p, q) = p + q \cdot t^2}$**,** where p and q are unknown parameters, to the data set $\{(t_k, y_k)\} = \{(-1, 1), (0, 5), (1, 3)\}$. Compute all **residuals**, and **plot** the model and the data points. Make sure to explain your solution method, very briefly.

8) (6pts) Consider the following MATLAB program. Write down the output value of variable **F** after the first **two iterations** of the for-loop, in terms of the values of **x** and **q**. What elementary function is this code computing?

```
function F = myFunction(x, q)
   F = 1;   term = 1;
   for k = 1 : 4
         term = term * q * x / k;
         F = F + term;
         q = q - 1;
   end
end
```