Math 340 • Fall 2026 • Homework #0 • Victor Matveev

This assignment checks basic Calculus pre-requisites for this course. Although it will not be collected, you should expect a quiz based on this material at the beginning of the semester.

1) Consider the linear approximation $\sqrt{1+x} \approx 1 + \frac{x}{2}$

a) Use it to estimate $\sqrt{5}$ without a calculator. Hint: write $5 = 4 + 1 = 4\left(1+\frac{1}{4}\right)$.

$$\sqrt{5} = \sqrt{4+1} = \sqrt{4\left(1+\frac{1}{4}\right)} = 2\sqrt{1+\frac{1}{4}} \approx 2\left(1+\frac{1}{8}\right) = 2+\frac{1}{4} = \mathbf{2.25}$$

b) Compare your approximation in part "a" with the exact value of $\sqrt{5}$.

Actual value: $\sqrt{5} \approx \mathbf{2.2361}$.

Absolute error of linear approximation: $\epsilon_{abs} = |2.25 - 2.2361| \approx \mathbf{0.014}$

Relative error of linear approximation: $\epsilon_{rel} = \frac{\epsilon_{abs}}{\sqrt{5}} \approx 0.0062 = \mathbf{0.62\%}$ (fairly accurate!)

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2) Consider the function $f(x) = x^{1/3}$:

a) Find the quadratic Taylor polynomial of $f(x) = x^{1/3}$ centered around $x_0 = 1$

$$\begin{cases} f(x) = x^{1/3} \\ f'(x) = \frac{1}{3}x^{-2/3} \\ f''(x) = -\frac{2}{9}x^{-5/3} \\ f'''(x) = \frac{10}{27}x^{-8/3} \end{cases} \Rightarrow \begin{cases} f(1) = 1 \\ f'(1) = \frac{1}{3} \\ \frac{f''(1)}{2!} = -\frac{1}{9} \\ \frac{f'''(c)}{3!} = \frac{5}{81c^{8/3}} \end{cases} \Rightarrow \boxed{f(x) = \underbrace{1 + \frac{x-1}{3} - \frac{(x-1)^2}{9}}_{P_2(x)} + \underbrace{\frac{5}{81c^{8/3}}(x-1)^3}_{R_2(x)}} \text{ where } c \in (x, 1)$$

b) Use your quadratic Taylor polynomial to estimate $f(0.5)$.

$$f(0.5) \approx P_2(0.5) = 1 + \frac{0.5-1}{3} - \frac{(0.5-1)^2}{9} = 1 - \frac{1}{6} - \frac{1}{36} = \frac{29}{36} = 0.80\overline{5}$$

c) Compute the absolute error of your estimate by comparing it with the exact value $f(0.5) = 2^{-1/3}$.

- Exact value: $f(0.5) = 2^{-1/3} \approx \mathbf{0.7937}$.
- Absolute error: $\epsilon_{abs} = |f(0.5) - P_2(0.5)| \approx \mathbf{0.012}$

d) Use the Taylor remainder formula to find the upper bound for the approximation error. Compare this bound with the actual error found in part "c".

The remainder $R_2(x) = \frac{5(x-1)^3}{81c^{8/3}}$ is the largest when $c \in (0.5, 1)$ is the smallest, so we set $c = 0.5$:

$$|R_2(0.5)| \le \frac{5 \cdot 0.5^3}{81(0.5)^{\frac{8}{3}}} = \frac{5}{81} 2^{\frac{8}{3}-3} = \frac{5}{81 \cdot 2^{\frac{1}{3}}} \approx \mathbf{0.049}$$

This agrees with the actual error result (in fact, the remainder is much smaller than this upper bound)

3) Find the quadratic Taylor polynomial for $f(x) = e^{\sin(x)}$ centered at $x_0 = 0$ by substituting the Taylor series for $\sin(x)$ into the Taylor series for $\exp(u)$ (see below), keeping only terms up to degree 2:

- $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$
- $\exp(u) = 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} + \cdots$

Check you Taylor polynomial by comparing its value with the exact value of $f(x)$ at $x = 0.2$.

$$f(x) = \exp(\sin x) \approx 1 + \sin x + \frac{\sin^2 x}{2!} + O(x^3)$$

$$= 1 + \left(x - \frac{x^3}{3!} + O(x^5)\right) + \frac{1}{2}\underbrace{\left(x - \frac{x^3}{3!} + O(x^5)\right)^2}_{x^2 - x^4/3 + \ldots} + O(x^3) = 1 + x + \frac{x^2}{2} + O(x^3)$$

Let's check the accuracy of this quadratic approximation:

$$f(0.5) = \exp(\sin(0.5)) \approx \mathbf{1.615}$$

$$P_2(0.5) = 1 + \frac{1}{2} + \frac{1}{8} = \frac{13}{8} = \mathbf{1.625} \quad \text{(quite accurate!)}$$

4) Find the seventh derivative $f^{(7)}(0)$ of function $f(x) = x\exp(-x^2)$ without repeated differentiation of $f(x)$.

Hint: use the Taylor expansion of $\exp(u)$ in the previous problem (with $u = -x^2$) , and identify the coefficient of term x^7 in the result.

$$f(x) = x\exp(-x^2) = x\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right) = x - x^3 + \frac{x^5}{2} - \frac{\mathbf{x^7}}{\mathbf{6}} + \cdots$$

Therefore

$$\frac{f^{(7)}(0)}{7!} = -\frac{1}{6}$$

And we obtain $f^{(7)}(0) = -\frac{7!}{6} = -840$ (now try obtaining this by direct differentiation!)

5) Consider the function $f(x) = \frac{1}{x}$. Given $a = -1$ and $b = 1$, we obtain:

$$\frac{f(b) - f(a)}{b - a} = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{1 - (-1)}{2} = 1$$

Does the Mean Value Theorem guarantee that there exists $c \in (-1, 1)$ where $f'(c) = 1$? Explain.

There is no value of *c* where the derivative equals 1 (or any positive number for that matter) since $f'(x) = -1/x^2 \leq 0$: the function is decreasing at each point of its domain! Therefore, there is no tangent line with the same slope as the slope of the secant line through $(-1, -1)$ and $(1, 1)$.

The Mean Value Theorem does not apply however, since the function is not differentiable and is discontinuous in the middle of the interval $(-1,1)$: there is a singularity at $x = 0$. Thus, this function does not satisfy the conditions of the MVT.