Math 340 • Fall 2026 • Homework #1 • Victor Matveev

1. (10pts) Consider the linear approximation $\sqrt[3]{1+x} \approx 1 + \frac{x}{3}$

a) Use it to estimate $\sqrt[3]{6}$ without a calculator. Hint: write $6 = 8 - 2 = 8\left(1 - \frac{1}{4}\right)$.

$$\sqrt[3]{6} = \sqrt[3]{8-2} = \sqrt[3]{8\left(1-\frac{1}{4}\right)} = 2 \cdot \sqrt[3]{1-\frac{1}{4}} \approx 2\left(1-\frac{1}{12}\right) = 2 - \frac{1}{6} = \frac{11}{6} = \mathbf{1.8333\ldots}$$

b) Compare your approximation in part "a" with the exact value of $\sqrt[3]{6}$.

- Exact: $\sqrt[3]{6} = \mathbf{1.81712\ldots}$
- Absolute error: $\epsilon_{\text{abs}} \equiv \left|\frac{11}{6} - \sqrt[3]{6}\right| \approx \mathbf{0.0162}$
- Relative error: $\epsilon_{rel} = \frac{\epsilon_{abs}}{\sqrt[3]{6}} \approx \mathbf{0.9\%}$

2. (35pts) Consider the function $f(x) = \ln(x)$:

a) Find the quadratic Taylor polynomial of $f(x)$ centered around $a = 1$. Start by computing the relevant derivatives of function $f(x)$ at $a = 1$.

$$\begin{cases} f(x) = \ln x \\ f'(x) = \frac{1}{x} \\ f''(x) = -\frac{1}{x^2} \\ f'''(x) = \frac{2}{x^3} \end{cases} \Rightarrow \begin{cases} f(1) = \ln(1) = 0 \\ f'(1) = 1 \\ \frac{f''(1)}{2!} = -\frac{1}{2} \\ \frac{f'''(c)}{3!} = \frac{2}{3!c^3} = \frac{1}{3c^3} \end{cases} \Rightarrow \boxed{f(x) = \underbrace{(x-1) - \frac{(x-1)^2}{2}}_{P_2(x)} + \underbrace{\frac{1}{3c^3}(x-1)^3}_{R_2(x)}} \text{ where } c \in (x, 1)$$

b) Use your quadratic Taylor polynomial to estimate $f(0.8)$:

$$f(0.8) \approx P_2(0.8) = (0.8-1) - \frac{(0.8-1)^2}{2} = -0.2 - \frac{0.04}{2} = \mathbf{-0.22}$$

c) Find an upper bound on the **remainder** of this quadratic polynomial approximation. Compare it with the **actual error** of your polynomial estimate of $f(0.8)$.

The remainder $R_2(x) = \frac{(x-1)^3}{3c^3}$ is the largest when $c \in (0.8, 1)$ is the smallest, so we set $c = 0.8$:

$$|R_2(0.8)| \le \frac{0.2^3}{3(0.8)^3} = \frac{1}{3 \cdot 4^3} = \frac{1}{192} \approx \mathbf{0.0052}$$

This agrees with the actual error:

- Exact value: $f(0.8) = \ln(0.8) \approx \mathbf{-0.223144}$.
- Actual absolute error: $\epsilon_{abs} = |f(0.8) - P_2(0.8)| \approx \mathbf{0.003}$

3. (20pts) Find the derivative $f^{(5)}(0)$ of function $f(x) = x\cos(2x)$ without repeated differentiation of $f(x)$.

Hint: use the Taylor expansion of $\cos(u)$ with $u = 2x$, and identify the coefficient of term $\boldsymbol{x^5}$ in the result.

$$f(x) = x\cos(2x) = x\left(1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{4!} + O(x^6)\right) = x - 2x^3 + \frac{2x^5}{3} + O(x^7)$$

Therefore $\frac{f^{(5)}(0)}{5!} = \frac{2}{3}$, and we obtain:

$$f^{(5)}(0) = \frac{2 \cdot 5!}{3} = 80$$

4. (35pts) Find the 4th degree Taylor polynomial for $f(x) = \sqrt{\cos x}$ centered at $x_0 = 0$ by combining the Taylor series for $\cos(x)$ with the series for $\sqrt{1+x}$ (see below), keeping only terms up to degree 4.

We will use the following series: $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^6)$, $\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} + O(x^3)$

$$f(x) = \sqrt{\cos(x)} = \sqrt{1 - \frac{x^2}{2} + \frac{x^4}{4!} + O(x^6)} = \sqrt{1 - \underbrace{\left(\frac{x^2}{2} - \frac{x^4}{4!} + O(x^6)\right)}_{Y}}$$

$$= 1 - \frac{Y}{2} - \frac{Y^2}{8} + O(Y^3) \quad \text{(this series was given in the table of common expansions)}$$

$$= 1 - \frac{1}{2}\left(\frac{x^2}{2} - \frac{x^4}{4!} + O(x^6)\right) - \frac{1}{8}\underbrace{\left(\frac{x^2}{2} - \frac{x^4}{4!} + O(x^6)\right)^2}_{\frac{x^4}{4} + O(x^6)} + \underbrace{O(Y^3)}_{O(x^6)}$$

$$= 1 - \frac{x^2}{4} + \underbrace{\left(\frac{1}{48} - \frac{1}{32}\right)}_{\frac{1}{16}\left(\frac{1}{3} - \frac{1}{2}\right) = -\frac{1}{96}} x^4 + O(x^6) = 1 - \frac{x^2}{4} - \frac{x^4}{96} + O(x^6)$$

Check your Taylor polynomial by comparing its value with the exact value of $f(x)$ at $x = 0.2$.

- Approximate value: $f(0.2) \approx 1 - \frac{0.2^2}{4} - \frac{0.2^4}{96} = 0.99 - \frac{0.0001}{6} \approx \mathbf{0.98998333}$
- Exact value: $f(0.2) = \sqrt{\cos 0.8} \approx \mathbf{0.98998312}$.
- Actual absolute error: $\epsilon_{abs} = |f(0.2) - P_2(0.2)| \approx \mathbf{0.0000002}$
- Actual relative error: $\epsilon_{rel} = \frac{\epsilon_{abs}}{f(0.2)} \approx \mathbf{2.2 \cdot 10^{-7}} = \mathbf{0.000022\%}$