Math 340 • Fall 2026 • Homework #3 • Victor Matveev

Problem 1 (40pts) Consider the root (zero) of function $f(x) = x^3 - x - 1$.

a) Find an obvious bracketing interval $[a, b]$ (with integer *a* and *b*) that contains the positive root of $f(x)$. Make a plot of $f(x)$ over this interval.

[1, 2] is an obvious bracketing interval: $f(1) = -1 < 0,\ f(2) = 8 - 2 - 1 = 5 > 0$

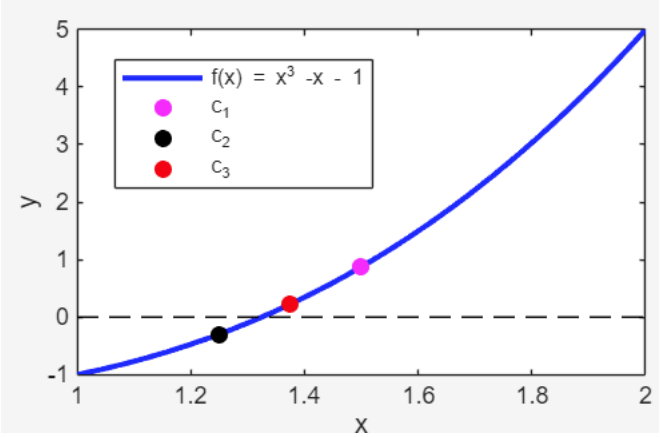
Note: [0, 2] also accepted as a correct interval.

b) Perform three iterations of the bisection method, "manually".

$$c_1 = \frac{a_1 + b_1}{2} = \frac{1+2}{2} = \frac{3}{2}: \quad f\left(\frac{3}{2}\right) = \frac{27}{8} - \frac{3}{2} - 1 = \frac{7}{8} > 0 \Rightarrow b_2 \doteq \frac{3}{2},\ a_2 = a_1$$

$$c_2 = \frac{a_2 + b_2}{2} = \frac{1+\frac{3}{2}}{2} = \frac{5}{4}: \quad f\left(\frac{5}{4}\right) = \frac{125}{64} - \frac{5}{4} - 1 = -\frac{19}{64} < 0 \Rightarrow a_3 \doteq \frac{5}{4},\ b_3 = b_2$$

$$c_3 = \frac{a_3 + b_3}{2} = \frac{\frac{5}{4}+\frac{3}{2}}{2} = \frac{11}{8}: \quad f\left(\frac{11}{8}\right) = \frac{1331}{512} - \frac{11}{8} - 1 = \frac{115}{512} > 0 \Rightarrow b_4 \doteq \frac{13}{8},\ a_4 = a_3$$



c) Find the number of bisection steps required to guarantee an absolute error of 10^{-9} or smaller (count the initial calculation $x_1 \equiv c_1 = \frac{a+b}{2}$ as the first iteration).

We need $\varepsilon_n \le \frac{b-a}{2^n} = \frac{2-1}{2^n} = \boxed{\frac{1}{2^n} \le 10^{-9}}$

Take logarithm with *any* base to solve this:

$\log_2$: $-n \le -9\log_2 10 \Rightarrow n \ge 9\log_2 10 \approx \boxed{30}$

$\log_{10}$: $-n\log_{10} 2 \le -9 \Rightarrow n \ge \frac{9}{\log_{10} 2} \approx \boxed{30}$

ln: $-n \ln 2 \le -9 \ln 10 \Rightarrow n \ge 9\frac{\ln 10}{\ln 2} \approx \boxed{30}$

d) Transform the root-finding problem $\boldsymbol{f(x)=0}$ to a fixed-point problem $\boldsymbol{x=g(x)\equiv x^3-1}$. Perform two iterations $x_{n+1}=g(x_n)$, starting with an initial guess $x_0=1$. Does this iteration converge? Explain.

$$f(x)=x^3-x-1=0 \;\Rightarrow\; x=g(x)=x^3-1 \;\Rightarrow\; \begin{cases} x_1=g(x_0)=g(1)=0 \\ x_2=g(x_1)=g(0)=-1 \\ x_3=g(x_2)=g(-1)=-2 \\ x_4=g(x_3)=g(-2)=-9 \end{cases}$$ Diverging!

Explanation: $|g'(r)|=3|r^2|$; In part (a) we found that $r>1$, therefore $|g'(r)|>1$

Problem 2 (60pts) Consider the following two functions $g(x)$:

$$(a)\;\; g(x)=x^2+\frac{3}{2}(1-x) \qquad (b)\;\; g(x)=\frac{3}{4}\left(x+\frac{1}{x}\right)$$

For each of these two functions, do the following:

i. Find all fixed points of the iteration $\boldsymbol{x_{n+1}=g(x_n)}$ (i.e. solve the equation $r=g(r)$).

$$(a)\; r=r^2+\frac{3}{2}(1-r) \;\Rightarrow\; 2r^2-5r+3=0 \;\Rightarrow\; r_{1,2}=\frac{5\pm\sqrt{25-24}}{4}=\frac{5\pm1}{4} \;\Rightarrow\; \begin{cases} r_1=1 \\ r_2=\frac{3}{2} \end{cases}$$

$$(b)\; r=\frac{3}{4}\left(r+\frac{1}{r}\right) \;\Rightarrow\; \frac{r}{4}=\frac{3}{4r} \;\Rightarrow\; r^2=3 \;\Rightarrow\; \begin{cases} r_1=+\sqrt{3} \\ r_2=-\sqrt{3} \end{cases}$$

ii. For each of the fixed points *r*, indicate whether the iteration would converge (at least locally). In other words, determine which of the fixed points are stable. To do that, determine whether the stability condition (contraction property) is satisfied:

$$C=|g'(r)|<1$$

$$(a)\; g'(r)=2r-\frac{3}{2} \;\Rightarrow\; \begin{cases} r_1=1:\; |g'(r_1)|=2-\frac{3}{2}=\mathbf{\frac{1}{2}}<1: \textbf{ Stable} \text{ (iteration converges locally to } r_1) \\ r_2=\frac{3}{2}:\; |g'(r_1)|=3-\frac{3}{2}=\mathbf{\frac{3}{2}}>1: \textbf{ Unstable } \text{(iteration does } \textbf{not} \text{ converge to } r_2) \end{cases}$$

$$(b)\; g'(r)=\frac{3}{4}\left(1-\frac{1}{r^2}\right) \;\Rightarrow\; g'(r_{1,2})=g'\left(\pm\sqrt{3}\right)=\frac{3}{4}\left(1-\frac{1}{3}\right)=\frac{1}{2}<1:$$ Both **stable** (iteration converges locally to $r_{1,2}$)

iii. For each stable fixed point, estimate the number of steps needed to achieve an absolute root error (tolerance) of 10^{-13}, assuming that the initial guess has an error of about 0.1.

Hint: $\epsilon_n \approx C^n \epsilon_0$

$$(\text{a})\ r_1 = 1:\ C \equiv \left|g'(r_1)\right| = \frac{1}{2} \Rightarrow \varepsilon_n \approx \varepsilon_0 C^n = \frac{0.1}{2^n} \le 10^{-13} \Rightarrow 2^n \ge 10^{12} \overset{\log_2}{\Rightarrow} n \ge 12\log_2 10 \approx \boxed{40}$$

$$(b)\ g'(r_{1,2}) = g'\left(\pm\sqrt{3}\right) = \frac{1}{2} \Rightarrow \text{same answer}\quad n \ge 12\log_2 10 \approx \boxed{40}$$

Review of bisection and direct iteration is here:
https://web.njit.edu/~matveev/Courses/M340_F26/M340_F26_Bisection_Iteration.pdf