Math 340 • Fall 2026 • Homework #4 • Victor Matveev

Sections 1.4-1.5: Newton's method and the Secant method.

Problem 1 (40pts) Consider the problem of finding the root (zero) of the function $\boldsymbol{f(x) = x - e^{-x}}$.

a) Set up the Newton's iterations for finding the root of $f(x)$, and compute x_1 and x_2, starting with $x_0 = 0$ (write down the **exact** expression for x_2, along with its decimal approximation to 6 digits).

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - e^{-x_n}}{1+e^{-x_n}} \Rightarrow \begin{cases} x_0 = 0, \\ x_1 = 0 - \dfrac{0-e^0}{1+e^0} = \dfrac{1}{2}, \\ x_2 = \dfrac{1}{2} - \dfrac{0.5 - e^{-1/2}}{1+e^{-1/2}} = \dfrac{3}{2\left(e^{1/2}+1\right)} \approx 0.566311. \end{cases}$$

b) Set up and perform **one** iteration of the Secant method to find x_2, starting with $x_0 = 0$ and $x_1 = 1$ (recall that two starting values are required).

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} = x_n - \left(x_n - e^{-x_n}\right)\frac{x_n - x_{n-1}}{x_n - e^{-x_n} - (x_{n-1} - e^{-x_{n-1}})}$$

$$\Rightarrow \begin{cases} x_0 = 0, \quad x_1 = 1, \\ x_2 = 1 - \left(1 - e^{-1}\right)\dfrac{1-0}{1-e^{-1} - \left(0 - e^0\right)} = 1 - \dfrac{1-e^{-1}}{2-e^{-1}} = \dfrac{1}{2-e^{-1}} \approx 0.612700. \end{cases}$$

What happens if you start with $x_0 = 1$ and $x_1 = 0$ instead?

We will obtain the same answer, of course, since it's the same secant line!

c) Which of the two values x_2 found in (a) and (b) is more accurate? Answer by comparing the so-called "residuals" for each method, $f(x_2)$ (i.e. compare how close the function value is to zero at x_2)

$$\begin{cases} \text{Newton's: } x_2 \approx 0.566311 \Rightarrow f(x_2) = x_2 - e^{-x_2} + x_2 - 2 \approx -0.0013 : \text{ much better!} \\ \text{Secant: } \quad x_2 \approx 0.612700 \Rightarrow f(x_2) = x_2 - e^{-x_2} + x_2 - 2 \approx -0.0708 \end{cases}$$

Problem 2 (60pts) Consider the problem of finding the principal value of $\boldsymbol{\arctan(A)}$, which is equivalent to the problem for finding the principal root of function $\boldsymbol{f(x) = \tan(x) - A}$, contained within interval $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

a) Set up the Newton's iteration for finding the root of the function $\boldsymbol{f(x) = \tan(x) - A}$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\tan(x_n) - A}{\sec^2(x_n)} = x_n - \cos^2(x_n)\left(\tan(x_n) - A\right) = x_n - \cos(x_n)\left(\sin(x_n) - A\cos(x_n)\right)$$

b) For $\boldsymbol{A = 2}$, perform 1 iteration of the Newton's method to find x_1, starting with $x_0 = 1$.

$$x_1 = 1 - \cos(1)\left(\sin(1) - 2\cos(1)\right) \approx 1.129204$$

c) Compute the absolute errors of your iterates x_0 and x_1 by comparing them to the true root:

$$\arctan 2 = 1.1071487177.$$

$$\varepsilon_0 = \left|x_0 - \tan^{-1}(2)\right| \approx \left|1.0000 - 1.10715\right| \approx 0.107$$

$$\varepsilon_1 = \left|x_1 - \tan^{-1}(2)\right| \approx \left|1.1292 - 1.10715\right| \approx 0.022$$

d) For Newton's method, the asymptotic error (i.e. error close to the root) satisfies

$$\epsilon_{n+1} \approx M\epsilon_n^2 \quad \text{where} \quad M = \frac{f''(r)}{2f'(r)}.$$

Given $\boldsymbol{f(x) = \tan(x) - A}$, simplify the above expression for M. For $A = 2$, compare $\epsilon_1 \approx M\epsilon_0^2$ with the actual error ϵ_1 found in part "c". How accurate is this approximation for the error?

$$\frac{f''(x)}{2f'(x)} = \frac{(\tan(x) - A)''}{2(\tan(x) - A)'} = \frac{\left(\sec^2(x)\right)'}{2\sec^2(x)} = \frac{2\sec(x)\cdot\sec(x)\tan(x)}{2\sec^2(x)} = \tan(x)$$

$$\Rightarrow \; M = \tan(r) = \tan\left(\tan^{-1}(A)\right) = A = 2 \;\; :)$$

Let's see how accurate this is:

$$\epsilon_1 \approx M\epsilon_0^2 = 2\epsilon_0^2 = 2(0.107)^2 \approx 0.023 \quad \textit{Very accurate!! :)}$$

e) Use your result in part "d" to estimate the number of steps required to reach the absolute tolerance $\epsilon_n \leq 2^{-53} \approx 10^{-16}$, given that $\epsilon_0 = 2^{-3} = 0.125$. Only 5 steps are needed to reach full machine precision:

$$\epsilon_1 \approx 2\epsilon_0^2 = 2 \cdot 2^{-6} = 2^{-5}$$

$$\epsilon_2 \approx 2\epsilon_1^2 = 2 \cdot 2^{-10} = 2^{-9}$$

$$\epsilon_3 \approx 2\epsilon_2^2 = 2 \cdot 2^{-18} = 2^{-17}$$

$$\epsilon_4 \approx 2\epsilon_3^2 = 2 \cdot 2^{-34} = 2^{-33}$$

$$\epsilon_5 \approx 2\epsilon_4^2 = 2 \cdot 2^{-66} = 2^{-65}$$