Math 340 • Fall 2026 • Homework #5 • Victor Matveev

Sections 1.4 and 4.1: Modified Newton's method; Least Squares.

Please show all your work.

Problem 1 (35pts)

To find roots of multiplicity $\boldsymbol{m} > 1$ (zero of order $\boldsymbol{m} > 1$), Newton's method has to be modified as follows:

$$\boldsymbol{x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}}$$

Consider the function $\boldsymbol{f(x) = \sin^3(x)}$, which has a zero of order $\boldsymbol{m = 3}$ at $\boldsymbol{r = \pi}$ (and any other integer multiple of π).

a) Plug in this function $f(x)$ into the above iteration: $x_{n+1} = x_n - \mathbf{3}\frac{f(x_n)}{f'(x_n)}$. Simplify this expression as much as possible.

b) Starting with $\boldsymbol{x_0 = 3}$, find the two iterates (x_1, x_2), recording each with **11** significant decimal digits of precision.

c) Compute the absolute error of each iterate: $\epsilon_0 = |x_0 - \pi|$, $\epsilon_1 = |x_1 - \pi|$ and $\epsilon_2 = |x_2 - \pi|$.

d) Use ϵ_0, ϵ_1 and ϵ_2 that you obtained to estimate convergence order ***p*** of this iteration, defined by the relationship $\epsilon_n \approx M\epsilon_{n-1}^p$, where $M = const$. Hint: given three values of ϵ_n, you can eliminate ***M*** and solve for ***p*** (use a logarithm).

Problem 2 (25pts) Suppose our data is described by an underlying linear model $\boldsymbol{y_{model}(t; p, q) = p \cdot t + q}$ with two unknown parameters $(\boldsymbol{p, q})$.

a) Find parameters $(\boldsymbol{p, q})$ minimizing sum of squared residuals for the data $\{(t_k, y_k)\} = \{(\mathbf{0,0}), (\mathbf{1,1}), (\mathbf{1,3}), (\mathbf{2,6})\}$.

b) **Plot** the data points and the best-fit (model) line. You may do it by hand, since it's just a line.

c) Compute the four residuals: $\boldsymbol{r_k = y_k^{data} - y_k^{model}}$.

Note: you may use the result derived in class: parameters minimizing $L(p,q) = \sum_{k=1}^{N}\left(y_{model}(t_k; p, q) - y_k\right)^2$ satisfy

$$\begin{pmatrix} \sum_{k=1}^{N} t_k^2 & \sum_{k=1}^{N} t_k \\ \sum_{k=1}^{N} t_k & \sum_{k=1}^{N} 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{N} t_k y_k \\ \sum_{k=1}^{N} y_k \end{pmatrix}$$

Problem 3 (40pts) Suppose the same data as in problem 2 is now described by a different underlying exponential model $\boldsymbol{y_{model}(t; p, q) = p \cdot 2^t + q}$, again with two unknown parameters $(\boldsymbol{p, q})$.

a) Show that minimizing the sum of squared residuals, $L(p,q) = \sum_{k=1}^{N=4}\left(y_{model}(t_k; p, q) - y_k\right)^2$ leads to the system

$$\begin{pmatrix} \sum_{k=1}^{N} 4^{t_k} & \sum_{k=1}^{N} 2^{t_k} \\ \sum_{k=1}^{N} 2^{t_k} & \sum_{k=1}^{N} 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{N} 2^{t_k} y_k \\ \sum_{k=1}^{N} y_k \end{pmatrix}$$

Hint: differentiate the *L* function above, setting $\frac{\partial L}{\partial p} = 0$, $\frac{\partial L}{\partial q} = 0$.

b) Solve this system to find model parameters $(\boldsymbol{p, q})$ given same data points $\{(t_k, y_k)\} = \{(\mathbf{0,0}), (\mathbf{1,1}), (\mathbf{1,3}), (\mathbf{2,6})\}$.

c) Plot the model and the data points, and compute the residuals.