Math 340 • Fall 2026 • Homework #5 • Victor Matveev

Sections 1.4 and 4.1: Modified Newton's method; Least Squares.

Please show all your work.

Problem 1 (35pts)

To find roots of multiplicity $m > 1$ (zero of order $m > 1$), Newton's method has to be modified as follows:

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$$

Consider the function $f(x) = \sin^3(x)$, which has a zero of order $m = 3$ at $r = \pi$ (and any other integer multiple of π).

a) Plug in this function $f(x)$ into the above iteration: $x_{n+1} = x_n - 3\frac{f(x_n)}{f'(x_n)}$. Simplify this expression as much as possible.

$$x_{n+1} = x_n - 3\frac{\sin^3(x_n)}{3\sin^2(x_n)\cos(x_n)} = \boxed{x_n - \tan(x_n)}$$

b) Starting with $x_0 = 3$, find the two iterates (x_1, x_2), recording each with **11** significant decimal digits of precision

c) Compute the absolute error of each iterate: $\epsilon_0 = |x_0 - \pi|$, $\epsilon_1 = |x_1 - \pi|$ and $\epsilon_2 = |x_2 - \pi|$.

$$\begin{cases} x_0 = 3 & \left(\text{error } \varepsilon_0 = |3-\pi| \approx 1.416\cdot 10^{-1}\right) \\ x_1 = 3-\tan(3) \approx \boxed{3.1425465431} & \left(\text{error } \varepsilon_1 = |x_1-\pi| \approx 9.539\cdot 10^{-4}\right) \\ x_2 = x_1-\tan(x_1) \approx \boxed{3.1415926533} & \left(\text{error } \varepsilon_2 = |x_2-\pi| \approx 2.893\cdot 10^{-10}\right) \end{cases}$$

d) Use ϵ_0, ϵ_1 and ϵ_2 that you obtained to estimate convergence order ***p*** of this iteration, defined by the relationship $\epsilon_n \approx M\epsilon_{n-1}^p$, where $M = const$. Hint: given three values of ϵ_n, you can eliminate ***M*** and solve for ***p*** (use a logarithm).

$$\begin{cases} \varepsilon_1 \approx M\varepsilon_0^p \\ \varepsilon_2 \approx M\varepsilon_1^p \end{cases} \xrightarrow{\div} \frac{\varepsilon_1}{\varepsilon_2} \approx \left(\frac{\varepsilon_0}{\varepsilon_1}\right)^p \xrightarrow{\log} \log\left(\frac{\varepsilon_1}{\varepsilon_2}\right) = p\log\left(\frac{\varepsilon_0}{\varepsilon_1}\right) \xrightarrow{\div} p = \frac{\log(\varepsilon_1/\varepsilon_2)}{\log(\varepsilon_0/\varepsilon_1)} \approx 3.0016$$

This numerical calculation strongly suggests 3rd order convergence: nice!! :)

If you are curious why it's 3rd order and not 2nd order, you can do a simple calculation / check:

$$g'(r) = 1-\sec^2(\pi) = 1-1 = 0 \Rightarrow \text{faster than linear convergence}$$
$$g''(r) = 2\sec^2(\pi)\tan(\pi) = 0 \Rightarrow \text{faster than } 2^{nd}\text{-order convergence}$$
$$g'''(r) = 4\sec^3(\pi)\tan^2(\pi) + 2\sec^4(\pi) = 2 \neq 0 \Rightarrow 3^{rd} \text{ order convergence}$$

Problem 2 (25pts) Suppose our data is described by an underlying linear model $\boldsymbol{y_{model}(t;p,q) = p \cdot t + q}$ with two unknown parameters $\boldsymbol{(p,q)}$.

a) Find parameters $\boldsymbol{(p,q)}$ minimizing sum of squared residuals for the data $\{(t_k, y_k)\} = \{\mathbf{(0,0),(1,1),(1,3),(2,6)}\}$.

$$\begin{cases} \sum_{k=1}^{N} t_k^2 = 0+1^2+1^2+2^2 = 6 \\ \sum_{k=1}^{N} t_k = 0+1+1+2 = 4 \\ \sum_{k=1}^{N} 1 = N = 4 \\ \sum_{k=1}^{N} t_k y_k = 0\cdot 0+1\cdot 1+1\cdot 3+2\cdot 6 = 16 \\ \sum_{k=1}^{N} y_k = 0+1+3+6 = 10 \end{cases} \Rightarrow \begin{pmatrix} \sum_{k=1}^{N} t_k^2 & \sum_{k=1}^{N} t_k \\ \sum_{k=1}^{N} t_k & \sum_{k=1}^{N} 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{N} t_k y_k \\ \sum_{k=1}^{N} y_k \end{pmatrix} \Rightarrow \begin{pmatrix} 6 & 4 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 16 \\ 10 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$$

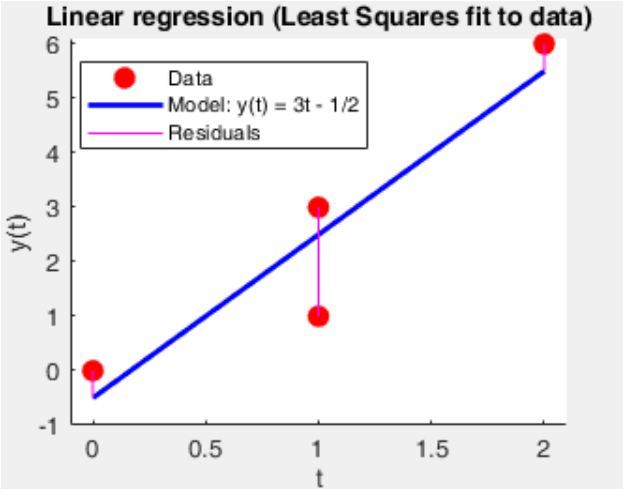
We need only 2 row operations to solve:

Start with $R_1 - R_2 \to R_1$: $\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$. Thus, we know that $\boldsymbol{p = 3}$

Now, perform $R_2 - 2R_1 \to R_2$: $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. Therefore, $\boldsymbol{q = -\frac{1}{2}}$

We conclude that the model underlying our data is $\boldsymbol{y_{model}(t) = 3t - \frac{1}{2}}$

b) **Plot** the data points and the best-fit (model) line. You may do it by hand, since it's just a line.



c) Compute the four residuals: $\boldsymbol{r_k = y_k^{model} - y_k^{data}}$:

$r_1 = -\frac{1}{2}$; $r_2 = \frac{5}{2} - 1 = \frac{3}{2}$; $r_3 = \frac{5}{2} - 3 = -\frac{1}{2}$; $r_4 = \frac{11}{2} - 6 = -\frac{1}{2}$.

Sum of residuals equals zero, since the model contains a constant term q (this will be explained in class)

$$\boldsymbol{RSS = \sum r_k^2 = \frac{1}{4} + \frac{9}{4} + \frac{1}{4} + \frac{1}{4} = 3}$$

Problem 3 (40pts) Suppose the same data as in problem 2 is now described by a different underlying exponential model $y_{model}(t;p,q) = p \cdot 2^t + q$, again with two unknown parameters (p,q).

a) Show that minimizing the sum of squared residuals, $L(p,q) = \sum_{k=1}^{N=4} \left(y_{model}(t_k;p,q) - y_k\right)^2$ leads to the system

$$\begin{pmatrix} \sum_{k=1}^{N} 4^{t_k} & \sum_{k=1}^{N} 2^{t_k} \\ \sum_{k=1}^{N} 2^{t_k} & \sum_{k=1}^{N} 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{N} 2^{t_k} y_k \\ \sum_{k=1}^{N} y_k \end{pmatrix}$$

Hint: differentiate the *L* function above, setting $\frac{\partial L}{\partial p} = 0,\ \frac{\partial L}{\partial q} = 0.$

The sum of squares function is: $L(p,q) \equiv \sum_{k=1}^{N} r_k^2 \equiv \sum_{k=1}^{N} \underbrace{\left[y_{model}(t_k;p,q) - y_{data}(t_k)\right]}_{\text{"Residuals" (measurement errors)}}^2 = \sum_{k=1}^{4} \left[p \cdot 2^{t_k} + q - y_k\right]^2$

Let's calculate partial derivatives with respect to parameters, and set them to zero:

$$\begin{cases} \frac{\partial L}{\partial p} = \frac{\partial}{\partial p} \sum_{k=1}^{N} \left[p \cdot 2^{t_k} + q - y_k\right]^2 = 2 \sum_{k=1}^{N} 2^{t_k} \left[p \cdot 2^{t_k} + q - y_k\right] = 2 \left(p \sum_{k=1}^{N} 4^{t_k} + q \sum_{k=1}^{N} 2^{t_k} - \sum_{k=1}^{N} 2^{t_k} y_k \right) = 0 \\ \frac{\partial L}{\partial q} = \frac{\partial}{\partial q} \sum_{k=1}^{N} \left[p \cdot 2^{t_k} + q - y_k\right]^2 = 2 \sum_{k=1}^{N} \left[p \cdot 2^{t_k} + q - y_k\right] \quad = 2 \left(p \sum_{k=1}^{N} 2^{t_k} + q \sum_{k=1}^{N} 1 - \sum_{k=1}^{N} y_k \right) = 0 \end{cases}$$

Here $\sum_{k=1..N} 1 = N$ (we keep the sum for pedagogical reasons). Cancelling the factor of 2, we obtain a linear system:

$$\begin{cases} p \sum_{k=1}^{N} 4^{t_k} + q \sum_{k=1}^{N} 2^{t_k} = \sum_{k=1}^{N} 2^{t_k} y_k \\ p \sum_{k=1}^{N} 2^{t_k} + q \sum_{k=1}^{N} 1 \quad = \sum_{k=1}^{N} y_k \end{cases}$$ which yields the expression above: $$\begin{pmatrix} \sum_{k=1}^{N} 4^{t_k} & \sum_{k=1}^{N} 2^{t_k} \\ \sum_{k=1}^{N} 2^{t_k} & \sum_{k=1}^{N} 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{N} 2^{t_k} y_k \\ \sum_{k=1}^{N} y_k \end{pmatrix}$$

b) Solve this system to find model parameters (p,q) given same data points $\{(t_k, y_k)\} = \{(0,0),(1,1),(1,3),(2,6)\}$.

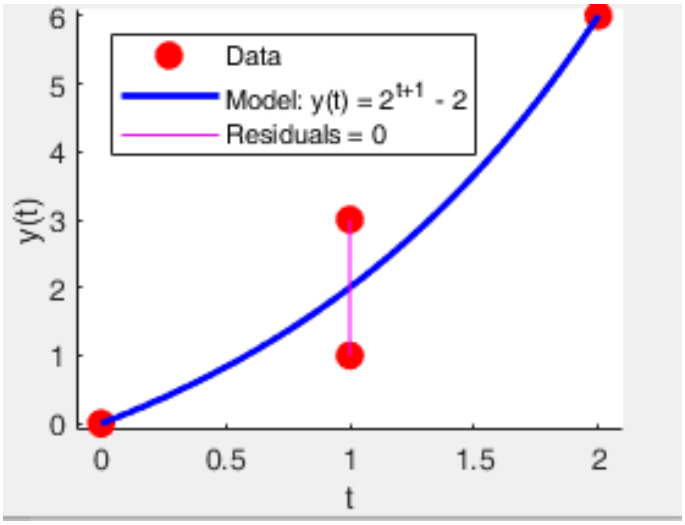
$$\begin{cases} \sum_{k=1}^{N} 4^{t_k} = 4^0 + 4^1 + 4^1 + 4^2 = 25 \\ \sum_{k=1}^{N} 2^{t_k} = 2^0 + 2^1 + 2^1 + 2^2 = 9 \end{cases} \quad \begin{cases} \sum_{k=1}^{N} 2^{t_k} y_k = 2^0 \cdot 0 + 2^1 \cdot 1 + 2^1 \cdot 3 + 2^2 \cdot 6 = 32 \\ \sum_{k=1}^{N} y_k = 0 + 1 + 3 + 6 = 10 \end{cases} \Rightarrow \begin{pmatrix} 25 & 9 \\ 9 & 4 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 32 \\ 10 \end{pmatrix}$$

- $25R_2 - 9R_1 \to R_2$: $\begin{pmatrix} 25 & 9 \\ 0 & 19 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 32 \\ 250 - 9 \cdot 32 \end{pmatrix} = \begin{pmatrix} 2 \\ -38 \end{pmatrix} \Rightarrow \begin{pmatrix} 25 & 9 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 32 \\ -2 \end{pmatrix}$. Thus, $\boldsymbol{q = -2}$

- $R_1 - R_2 \to R_1$: $\begin{pmatrix} 25 & 9 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 32 \\ -2 \end{pmatrix} \Rightarrow \begin{pmatrix} 25 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 50 \\ -2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$. Thus, $\boldsymbol{p = 2}$

We conclude that the model underlying our data is $\boldsymbol{y_{model}(t) = 2 \cdot 2^t - 2 = 2^{t+1} - 2}$ **.**

c) Plot the model and the data points, and compute the residuals.



Compute the four residuals: $\boldsymbol{r_k = y_k^{model} - y_k^{data}}$:

$r_1 = \mathbf{0}$; $r_2 = 2 - 1 = \mathbf{1}$; $r_3 = 2 - 3 = \mathbf{-1}$; $r_4 = 8 - 2 - 6 = \mathbf{0}$

Sum of residuals equals zero, since the model contains a constant term *q* (this will be explained in class)

$$\boldsymbol{RSS = \sum r_k^2 = \ 0 + 1 + 1 + 0 = 2}$$