Math 340 • Fall 2026 • Homework #6 • Victor Matveev

Sections 4.1-4.2: Parameter Estimation using Least Squares.

Please show all your work. See lecture notes: [Least Squares]

Problem 1 (35pts) (problem 4.1.10 from the textbook, page 206):

Given data points $\{(x_k, y_k, z_k)\} = \{\ \mathbf{(0,0,1), (0,1,2), (1,0,2), (1,1,5), (1,2,5)}\ \}$, find the plane

$$\boldsymbol{z = c_0 + c_1 x + c_2 y}$$

that best fits the data, in the least-squares sense. Compute also the **residuals** for this best fit.

Explain clearly your solution method. You may use either:

- The **geometric (orthogonality)** approach , or
- The **analytic (calculus)** approach (setting partial derivatives of sum of squared errors to zero, as in homework #5)

Hint: this problem is a straightforward example of **linear regression**, with

$$\boldsymbol{z_{model}(x, y; c_0, c_1, c_2) = c_0 + c_1 x + c_2 y}$$

You will obtain a 3x3 linear equation of form $\boldsymbol{Mx = b}$, where $x = [c_0, c_1, c_2]$ is a (column) vector of unknown best-fit parameters. The system you obtained is easily solvable using a couple row operations "by hand".

Problem 2 (35pts) (problem 4.2.6(b) on p. 218; see example 4.11 on page 216):

Suppose the amount of medication in the bloodstream is measured as a function of time after intake, giving the data set

$$\{(t_k, y_k)\} = \{\ \mathbf{(1,16), (2,4), (3,3), (4,8)}\ \}.$$

The amount of medication is modeled by

$$\boldsymbol{y_{model}(t; c, \alpha) = c\ t\ e^{\alpha t}}.$$

a) **Linearize** the model by taking the logarithm, and find the **best-fit** parameters c and α, in the least-squares sense.

b) **Plot** the best-fit model $y_{model}(t; c, \alpha) = c\ t\ e^{\alpha t}$ and the data on the same graph together with the data points on the same graph (you may use MATLAB, Python, or any computational tool of your choice).

Hint: in class we learned that this model can be linearized with respect to parameters using a logarithmic transformation

$$z_{model}(t) \equiv \ln(y_{model}) - \ln(t) = \alpha \cdot t + \ln c$$

leading to a **linear regression problem** with slope $p \equiv \alpha$ and intercept $q \equiv \ln$ c.

Problem 3 (30pts) Let's extend problem 2: suppose the data in that problem is better described by a model

$$\boldsymbol{y_{model}(t; c, p, \alpha) = c\ t^p\ e^{\alpha t}},$$

Which introduces an additional unknown parameter $\boldsymbol{p}$. Perform a **logarithmic transformation** to arrive at a linear relationship among the transformed variables and parameters.

Then, set up the **linear algebraic equations** ("normal equations") for the unknown parameters (c, α, p) obtained by applying least-squares method to a general data set $\{(t_k, y_k)\}$ (with N points). Finally, plug in the four data points from Problem 2 and write down the resulting numerical 3×3 system. You do **not** need to solve it: solving this system will be part of Lab #5.