Math 340 • Fall 2026 • Homework #6 • Victor Matveev

Sections 4.1-4.2: Parameter Estimation using Least Squares.

Please show all your work.

Problem 1 (35pts) (problem 4.1.10 from the textbook, page 206):

Given data points $\{(x_k, y_k, z_k)\} = \{\ (\mathbf{0},\mathbf{0},\mathbf{1}), (\mathbf{0},\mathbf{1},\mathbf{2}), (\mathbf{1},\mathbf{0},\mathbf{2}), (\mathbf{1},\mathbf{1},\mathbf{5}), (\mathbf{1},\mathbf{2},\mathbf{5})\ \}$, find the plane

$$z = c_0 + c_1 x + c_2 y$$

that best fits the data, in the least-squares sense. Compute also the **residuals** for this best fit.

Explain clearly your solution method. You may use either:

- The **geometric (orthogonality)** approach , or
- The **analytic (calculus)** approach (setting partial derivatives of sum of squared errors to zero, as in homework #5)

Hint: this problem is a straightforward example of **linear regression**, with $z_{model}(x, y; c_0, c_1, c_2) = c_0 + c_1 x + c_2 y$

The sum of squares function is: $L(c_{0,1,2}) \equiv \sum_{k=1}^{N} r_k^2 \equiv \sum_{k=1}^{N} \underbrace{\left[z_{\text{model}}(t_k; c_{0,1,2}) - z_{\text{data}}(t_k) \right]}_{\text{"Residuals" (model-data errors)}}^2 = \sum_{k=1}^{N} \left[c_0 + c_1 x_k + c_2 y_k - z_k \right]^2$

Let's do the calculus approach: set partial derivatives with respect to parameters to zero:

$$\begin{cases} \dfrac{\partial L}{\partial c_0} = \dfrac{\partial}{\partial c_0} \sum_{k=1}^{N} \left[c_0 + c_1 x_k + c_2 y_k - z_k \right]^2 = 2 \sum_{k=1}^{N} \left[c_0 + c_1 x_k + c_2 y_k - z_k \right] = 2 \left(c_0 \sum_{k=1}^{N} 1 + c_1 \sum_{k=1}^{N} x_k + c_2 \sum_{k=1}^{N} y_k - \sum_{k=1}^{N} z_k \right) = 0 \\ \dfrac{\partial L}{\partial c_1} = \dfrac{\partial}{\partial c_1} \sum_{k=1}^{N} \left[c_0 + c_1 x_k + c_2 y_k - z_k \right]^2 = 2 \sum_{k=1}^{N} x_k \left[c_0 + c_1 x_k + c_2 y_k - z_k \right] = 2 \left(c_0 \sum_{k=1}^{N} x_k + c_1 \sum_{k=1}^{N} x_k^2 + c_2 \sum_{k=1}^{N} x_k y_k - \sum_{k=1}^{N} x_k z_k \right) = 0 \\ \dfrac{\partial L}{\partial c_2} = \dfrac{\partial}{\partial c_2} \sum_{k=1}^{N} \left[c_0 + c_1 x_k + c_2 y_k - z_k \right]^2 = 2 \sum_{k=1}^{N} y_k \left[c_0 + c_1 x_k + c_2 y_k - z_k \right] = 2 \left(c_0 \sum_{k=1}^{N} y_k + c_1 \sum_{k=1}^{N} x_k y_k + c_2 \sum_{k=1}^{N} y_k^2 - \sum_{k=1}^{N} y_k z_k \right) = 0 \end{cases}$$

We obtain a linear system:

$$\begin{cases} c_0 \sum_{k=1}^{N} 1 + c_1 \sum_{k=1}^{N} x_k + c_2 \sum_{k=1}^{N} y_k = \sum_{k=1}^{N} z_k \\ c_0 \sum_{k=1}^{N} x_k + c_1 \sum_{k=1}^{N} x_k^2 + c_2 \sum_{k=1}^{N} x_k y_k = \sum_{k=1}^{N} x_k z_k \\ c_0 \sum_{k=1}^{N} y_k + c_1 \sum_{k=1}^{N} x_k y_k + c_2 \sum_{k=1}^{N} y_k^2 = \sum_{k=1}^{N} y_k z_k \end{cases} \quad \text{or, in matrix form:} \quad \begin{pmatrix} \sum_{k=1}^{N} 1 & \sum_{k=1}^{N} x_k & \sum_{k=1}^{N} y_k \\ \sum_{k=1}^{N} x_k & \sum_{k=1}^{N} x_k^2 & \sum_{k=1}^{N} x_k y_k \\ \sum_{k=1}^{N} y_k & \sum_{k=1}^{N} x_k y_k & \sum_{k=1}^{N} y_k^2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{N} z_k \\ \sum_{k=1}^{N} x_k z_k \\ \sum_{k=1}^{N} y_k z_k \end{pmatrix}$$

Now plug in data points $\{(x_k, y_k, z_k)\} = \{(\mathbf{0},\mathbf{0},\mathbf{1}), (\mathbf{0},\mathbf{1},\mathbf{2}), (\mathbf{1},\mathbf{0},\mathbf{2}), (\mathbf{1},\mathbf{1},\mathbf{5}), (\mathbf{1},\mathbf{2},\mathbf{5})\}$, and do 4 or 5 row operations:

$$\begin{pmatrix} 5 & 3 & 4 \\ 3 & 3 & 3 \\ 4 & 3 & 6 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 12 \\ 17 \end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 1 & 1 & 1 & 4 \\ 1 & 0 & 3 & 5 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 0 & 0 & -5 & -7 \\ 0 & 1 & -2 & -1 \\ 1 & 0 & 3 & 5 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 0 & 0 & 1 & 7/5 \\ 0 & 1 & -2 & -1 \\ 1 & 0 & 3 & 5 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 0 & 0 & 1 & 7/5 \\ 0 & 1 & 0 & 9/5 \\ 1 & 0 & 0 & 4/5 \end{array}\right) \Rightarrow \boxed{z_{\text{mod}} = \frac{4 + 9x + 7y}{5}}$$

Now we can compute the residuals, $r_k = z_{model}(x_k, y_k) - z_{data}(x_k, y_k)$:

$$r_1 = \frac{4}{5} - 1 = -\frac{\mathbf{1}}{\mathbf{5}}; \quad r_2 = \frac{11}{5} - 2 = \frac{\mathbf{1}}{\mathbf{5}}; \quad r_3 = \frac{13}{5} - 2 = \frac{\mathbf{3}}{\mathbf{5}}; \quad r_4 = \frac{20}{5} - 5 = -\mathbf{1}; \quad r_5 = \frac{27}{5} - 5 = \frac{\mathbf{2}}{\mathbf{5}}$$

Note: if one model component is a constant, then the sum of all residuals is always zero (this can be used as a check).

Problem 2 (35pts) (problem 4.2.6(b) on p. 218; see example 4.11 on page 216):

Suppose the amount of medication in the bloodstream is measured as a function of time after intake, giving the data set

$$\{(t_k, y_k)\} = \{\ (\mathbf{1}, \mathbf{16}), (\mathbf{2}, \mathbf{4}), (\mathbf{3}, \mathbf{3}), (\mathbf{4}, \mathbf{8})\ \}.$$

The amount of medication is modeled by

$$\boldsymbol{y_{model}(t; c, \alpha) = c\, t\, e^{\alpha t}}.$$

a) **Linearize** the model by taking the logarithm, and find the **best-fit** parameters *c* and α, in the least-squares sense.

b) **Plot** the best-fit model $y_{model}(t; c, \alpha) = c\ t\ e^{\alpha t}$ and the data on the same graph together with the data points on the same graph (you may use MATLAB, Python, or any computational tool of your choice).

Hint: in class we learned that this model can be linearized with respect to parameters using a logarithmic transformation

$$z_{model}(t) \equiv \ln(y_{model}) - \ln(t) = \alpha \cdot t + \ln c$$

leading to a **linear regression problem** with slope $\boldsymbol{p \equiv \alpha}$ and intercept $\boldsymbol{q \equiv \ln c}$.

Compute $z_k \equiv \ln(y_k) - \ln(t_k)$ for our data points $\{(t_k, y_k)\} = \{\ (\mathbf{1}, \mathbf{16}), (\mathbf{2}, \mathbf{4}), (\mathbf{3}, \mathbf{3}), (\mathbf{4}, \mathbf{8})\ \}$:

$$\{z_k\} = \{4\ln(2),\quad \ln(2),\quad 0,\quad \ln(2)\}$$

Now, perform linear regression:

$$\begin{cases} \sum_{k=1}^{N} t_k^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30, \quad \sum_{k=1}^{N} t_k = 1+2+3+4 = 10 \\ \sum_{k=1}^{N} t_k z_k = 4\ln(2) + 2\ln(2) + 4\ln(2) = 10\ln(2) \\ \sum_{k=1}^{N} z_k = 4\ln(2) + \ln(2) + \ln(2) = 6\ln(2) \end{cases} \Rightarrow \begin{pmatrix} \sum_{k=1}^{N} t_k^2 & \sum_{k=1}^{N} t_k \\ \sum_{k=1}^{N} t_k & \sum_{k=1}^{N} 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{N} t_k z_k \\ \sum_{k=1}^{N} z_k \end{pmatrix} \Rightarrow \begin{pmatrix} 30 & 10 \\ 10 & 4 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 10\ln(2) \\ 6\ln(2) \end{pmatrix} = \ln(2) \begin{pmatrix} 10 \\ 6 \end{pmatrix}$$

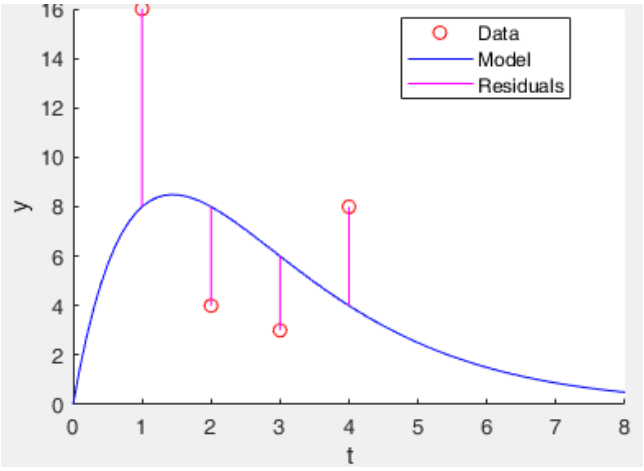
We need a couple row operations: $\frac{R_1}{10} \to R_1$, $\frac{R_2}{2} \to R_2$; $2R_2 - R_1 \to R_1$; $R_2 - 5R_1 \to R_2$

$$\begin{pmatrix} 30 & 10 \\ 10 & 4 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \ln(2) \begin{pmatrix} 10 \\ 6 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \ln(2) \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \ln(2) \begin{pmatrix} -1 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \ln(2) \begin{pmatrix} -1 \\ 8 \end{pmatrix}.$$

Thus, $p \equiv \alpha = -\ln 2$, $q = \ln c = 4\ln 2$. Now, find *c*: $\boldsymbol{c = e^q = e^{4\ln(2)} = 2^4 = 16}$

We obtain the following model: $y_{model}(t; c, \alpha) = cte^{\alpha t} = 16\ t\ e^{-\ln(2)t} = 16\ t\ 2^{-t}$

Plot shows that either the data quality is quite poor, or the model is not quite accurate for this particular drug:



Problem 3 (30pts) Extension of problem 2.

Suppose the data in problem 2 is better described by a model

$$y_{model}(t; c, p, \alpha) = c\, t^p\, e^{\alpha t},$$

Which introduces an additional unknown parameter p

Set up (but do not solve) the **linear algebraic equations** for the unknown parameters (c, α, p), given a general data set $\{(t_k, y_k)\}$ (with N points). Start with a **logarithmic transformation** to obtain a linear relationship among the transformed variables and parameters.

Transform the model first: $\boldsymbol{z_{model}(t) \equiv \ln(y_{model}) = \ln c + p \cdot \ln t + \alpha \cdot t}$

Denoting $q \equiv \ln c$, we notice that $\boldsymbol{z}$ is linear in parameters, with basis functions $\boldsymbol{\phi_0(t) = 1, \phi_1(t) = \ln t,}$ and $\boldsymbol{\phi_2(t) = t}$:

$$\boldsymbol{z_{model}(t) \equiv q \cdot \phi_0(t) + p \cdot \phi_1(t) + \alpha \cdot \phi_2(t)}$$

[Side note: the ordering of these functions is unimportant: our ordering here is different from the one in problem 2]

The general system for three basis functions (three parameters case) is

$$\underbrace{\begin{pmatrix} \sum_{k=1}^{N}\phi_0^2(t_k) & \sum_{k=1}^{N}\phi_0(t_k)\phi_1(t_k) & \sum_{k=1}^{N}\phi_0(t_k)\phi_2(t_k) \\ \sum_{k=1}^{N}\phi_1(t_k)\phi_0(t_k) & \sum_{k=1}^{N}\phi_1^2(t_k) & \sum_{k=1}^{N}\phi_1(t_k)\phi_2(t_k) \\ \sum_{k=1}^{N}\phi_2(t_k)\phi_0(t_k) & \sum_{k=1}^{N}\phi_2(t_k)\phi_1(t_k) & \sum_{k=1}^{N}\phi_2^2(t_k) \end{pmatrix}}_{A^T A x} \begin{pmatrix} q \\ p \\ \alpha \end{pmatrix} = \underbrace{\begin{pmatrix} \sum_{k=1}^{N}\phi_0(t_k) z_k \\ \sum_{k=1}^{N}\phi_1(t_k) z_k \\ \sum_{k=1}^{N}\phi_2(t_k) z_k \end{pmatrix}}_{Az}$$

In our case, this becomes $\begin{pmatrix} \sum_{k=1}^{N}1 & \sum_{k=1}^{N}\ln t_k & \sum_{k=1}^{N}t_k \\ \sum_{k=1}^{N}\ln t_k & \sum_{k=1}^{N}(\ln t_k)^2 & \sum_{k=1}^{N}t_k \ln t_k \\ \sum_{k=1}^{N}t_k & \sum_{k=1}^{N}t_k \ln t_k & \sum_{k=1}^{N}t_k^2 \end{pmatrix} \begin{pmatrix} q \\ p \\ \alpha \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{N}z_k \\ \sum_{k=1}^{N}z_k \ln t_k \\ \sum_{k=1}^{N}z_k t_k \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{N}\ln y_k \\ \sum_{k=1}^{N}\ln y_k \cdot \ln t_k \\ \sum_{k=1}^{N}\ln y_k \cdot t_k \end{pmatrix}$

In lab #6 you will code and solve this system for a given data set.

For the data set of problem 2, which is $\{(t_k, y_k)\} = \{\, (\mathbf{1}, \mathbf{16}), (\mathbf{2}, \mathbf{4}), (\mathbf{3}, \mathbf{3}), (\mathbf{4}, \mathbf{8}) \,\}$, this system becomes

$$\begin{pmatrix} 4 & 3\ln 2 + \ln 3 & 10 \\ 3\ln 2 + \ln 3 & 5(\ln 2)^2 + (\ln 3)^2 & 10\ln 2 + 3\ln 3 \\ 10 & 10\ln 2 + 3\ln 3 & 30 \end{pmatrix} \begin{pmatrix} q \\ p \\ \alpha \end{pmatrix} = \begin{pmatrix} 9\ln 2 + \ln 3 \\ 8(\ln 2)^2 + (\ln 3)^2 \\ 20\ln 2 + 3\ln 3 \end{pmatrix}$$

NOTE: Solving this system would give you a negative value of p, indicating either poor data quality, or poor model choice. This is also the case in problem 2.