Math 340 • Fall 2026 • Homework #8 • Victor Matveev

Section 3.1-3.3: Polynomial Interpolation, error bounds, and Chebyshev nodes

Please show all your work.

Problem 1 (20pts) Equidistant interpolation

Consider the quadratic interpolant $P_2(x)$ of $f(x)=\frac{1}{1+x}$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$ using the equidistant nodes $\{\boldsymbol{x_k}\}=\left\{-\frac{1}{2},\ \mathbf{0},+\frac{1}{2}\right\}$, obtained in Problem 2 of homework #7. Examine its error $R_2(x)$ (see last graph in hw #7 solution) in two different ways:

a) Evaluate the absolute interpolation error $|R_2(x)|\equiv|f(x)-P_2(x)|$ halfway between interpolation nodes, at $x=\pm\frac{1}{4}$.

b) Find an upper bound on the maximum interpolation error using:

$$\|R_2\|_\infty \equiv \max_{x\in\left[-\frac{1}{2},\frac{1}{2}\right]}|R_2(x)| \le \max_{c\in\left[-\frac{1}{2},\frac{1}{2}\right]}\left|\frac{f'''(c)}{3!}\right|\cdot\max_{x\in\left[-\frac{1}{2},\frac{1}{2}\right]}\left|(x-x_1)(x-x_2)(x-x_3)\right|$$

Hint: find the maximum of $|\psi(x)|\equiv|(x-x_1)(x-x_2)(x-x_3)|$ using the first derivative test.

Problem 2 (40pts) Chebyshev Polynomials

a) Derive the 4th degree Chebyshev polynomial, $T_4(x)$, using the recurrence relationship:

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x)$$

where $T_0(x)=1,\ T_1(x)=x$.

Verify your result independently by applying the double-angle formula twice to $T_4(x)=\cos\left(4\cos^{-1}(x)\right)$

b) Find the four Chebyshev nodes on [-1, 1], $x_k=\cos\left(\left(k-\frac{1}{2}\right)\cdot\frac{\pi}{n}\right)$, $k=1..n$ (where $n=4$). Express these nodes in radicals and verify that $T_4(x_k)=0$. **Hint:** use the half-angle formula.

c) Transform these four nodes from the interval [-1, 1] to the interval [0, 6].

Problem 3 (40pts) Chebyshev Interpolation

Now consider the same function and interval, but use the three Chebyshev nodes: $\{\boldsymbol{x_k}\}=\left\{-\frac{\sqrt{3}}{4},\ \mathbf{0},\frac{\sqrt{3}}{4}\right\}$:

a) Find the quadratic interpolating polynomial $P_2(x)$ for this set of nodes.

b) Plot $f(x)$ and $P_2(x)$ together on $x\in\left[-\frac{1}{2},\frac{1}{2}\right]$ (use any computational tool). On a separate graph, plot the error $R_2(x)=f(x)-P_2(x)$, on the same interval.

c) Find an upper bound on the maximum absolute interpolation error, using the remainder formula from problem 1(b).

Hint: for Chebyshev nodes on [a, b], we've proven in class that $\max_{x\in[a,b]}\left|(x-x_1)\ldots(x-x_n)\right|=\frac{h^n}{2^{n-1}}$, where $h=\frac{b-a}{2}$.

d) Compare the error bounds obtained in Problems 1(b) and 3(c). By what factor does the bound improve when equidistant nodes are replaced by Chebyshev nodes?