Math 340 • Fall 2026 • Homework #8 • Victor Matveev

Section 3.1-3.3: Polynomial Interpolation, error bounds, and Chebyshev nodes

Problem 1 (20pts) Equidistant interpolation

Consider the quadratic interpolant $P_2(x)$ of $f(x)=\frac{1}{1+x}$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$ using the equidistant nodes $\{x_k\}=\left\{-\frac{1}{2},\ 0,+\frac{1}{2}\right\}$, obtained in Problem 2 of homework #7. Examine its error $R_2(x)$ (see last graph in hw #7 solution) in two different ways:

a) Evaluate the absolute interpolation error $|R_2(x)|\equiv|f(x)-P_2(x)|$ halfway between interpolation nodes, at $x=\pm\frac{1}{4}$.

Recall that we found $P_2(x)=1-\frac{4}{3}x+\frac{4}{3}x^2$, therefore:

$$\left|R_2\left(\frac{1}{4}\right)\right|\equiv\left|f\left(\frac{1}{4}\right)-P_2\left(\frac{1}{4}\right)\right|=\left|\frac{1}{1+\frac{1}{4}}-\left(1-\frac{4}{3}\cdot\frac{1}{4}+\frac{4}{3}\cdot\left(\frac{1}{4}\right)^2\right)\right|=\left|\frac{4}{5}-\left(1-\frac{1}{3}+\frac{1}{12}\right)\right|=\left|\frac{4}{5}-\frac{3}{4}\right|=\frac{1}{20}=\mathbf{0.05}$$

$$\left|R_2\left(-\frac{1}{4}\right)\right|\equiv\left|f\left(-\frac{1}{4}\right)-P_2\left(-\frac{1}{4}\right)\right|=\left|\frac{1}{1-\frac{1}{4}}-\left(1+\frac{4}{3}\cdot\frac{1}{4}+\frac{4}{3}\cdot\left(\frac{1}{4}\right)^2\right)\right|=\left|\frac{4}{3}-\left(1+\frac{1}{3}+\frac{1}{12}\right)\right|=\left|\frac{4}{3}-\frac{17}{12}\right|=\frac{1}{12}\approx\mathbf{0.083}$$

Note that the 2nd value is very close to the actual max error shown in the graph for Problem 2 of Homework #7:

$$\left|\left|R_2\right|\right|_\infty\approx\mathbf{0.83}$$

b) Find an upper bound on the maximum interpolation error using:

$$\|R_2\|_\infty\equiv\max_{x\in\left[-\frac{1}{2},\frac{1}{2}\right]}|R_2(x)|\le\max_{c\in\left[-\frac{1}{2},\frac{1}{2}\right]}\left|\frac{f'''(c)}{3!}\right|\cdot\max_{x\in\left[-\frac{1}{2},\frac{1}{2}\right]}\left|(x-x_1)(x-x_2)(x-x_3)\right|$$

Hint: find the maximum of $|\psi(x)|\equiv|(x-x_1)(x-x_2)(x-x_3)|$ using the first derivative test.

$$\max_{c\in\left[-\frac{1}{2},\frac{1}{2}\right]}\left|\frac{f'''(c)}{3!}\right|=\max_{c\in\left[-\frac{1}{2},\frac{1}{2}\right]}\frac{1}{(1+c)^4}=\frac{1}{\left(1-\frac{1}{2}\right)^4}=\mathbf{16}$$

$$\psi(x)=(x-x_1)(x-x_2)(x-x_3)=\left(x+\frac{1}{2}\right)x\left(x-\frac{1}{2}\right)=x\left(x^2-\frac{1}{4}\right)=x^3-\frac{x}{4}$$

Critical points: $\psi'(x)=3x^2-\frac{1}{4}=0\ \Rightarrow\ x=\pm\frac{1}{2\sqrt{3}}$

$$\left|\psi\left(\pm\frac{1}{2\sqrt{3}}\right)\right|=\left|\pm\frac{1}{2\sqrt{3}}\left(\frac{1}{12}-\frac{1}{4}\right)\right|=\frac{1}{8\sqrt{3}}\cdot\frac{2}{3}=\mathbf{\frac{1}{12\sqrt{3}}}$$

Therefore, we obtain $\|R_2\|_\infty\le\frac{16}{12\sqrt{3}}=\frac{4}{3\sqrt{3}}\approx\mathbf{0.77}$

We of course obtain an over-estimate of the true max error using this error bound procedure.

Problem 2 (40pts) Chebyshev Polynomials

a) Derive the 4th degree Chebyshev polynomial, $T_4(x)$, using the recurrence relationship:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

where $T_0(x) = 1,\ T_1(x) = x.$

$$T_2(x) = 2xT_1(x) - T_0(x) = \mathbf{2x^2 - 1}$$

$$T_3(x) = 2xT_2(x) - T_1(x) = 2x(2x^2 - 1) - x = \mathbf{4x^3 - 3x}$$

$$T_4(x) = 2xT_3(x) - T_2(x) = 2x(4x^3 - 3x) - (2x^2 - 1) = 8x^4 - 8x^2 + 1$$

Verify your result independently by applying the double-angle formula twice to $T_4(x) = \cos\left(4\cos^{-1}(x)\right)$

Let's denote $\theta \equiv \cos^{-1} x$:

$$T_4(x) = \cos(4\theta) = 2\cos^2(2\theta) - 1 = 2(2\cos^2\theta - 1)^2 - 1 = 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1$$
$$= 8\cos^4\theta - 8\cos^2\theta + 1 = 8x^4 - 8x^2 + 1$$

b) Find the four Chebyshev nodes on [-1, 1], $x_k = \cos\left(\left(k - \frac{1}{2}\right)\cdot\frac{\pi}{n}\right),\ k = 1..n$ (where $n = 4$). Express these nodes in radicals and verify that $T_4(x_k) = 0$. **Hint:** use the half-angle formula.

$$x_1 = \cos\left(\frac{1}{2}\cdot\frac{\pi}{4}\right) = \sqrt{\frac{1 + \cos\frac{\pi}{4}}{2}} = \frac{\sqrt{\mathbf{2} + \sqrt{\mathbf{2}}}}{\mathbf{2}} \approx 0.92388$$

$$x_2 = \cos\left(\frac{3}{2}\cdot\frac{\pi}{4}\right) = \sqrt{\frac{1 + \cos\frac{3\pi}{4}}{2}} = \frac{\sqrt{\mathbf{2} - \sqrt{\mathbf{2}}}}{\mathbf{2}} \approx 0.38268$$

The nodes are symmetric with respect to the mid-point at $x = 0$, therefore:

$$x_3 = \cos\left(\frac{5}{2}\cdot\frac{\pi}{4}\right) = \cos\left(\frac{5\pi}{8}\right) = -\cos\left(\frac{3\pi}{8}\right) = -\frac{\sqrt{\mathbf{2} - \sqrt{\mathbf{2}}}}{\mathbf{2}} \approx -0.38268$$

$$x_4 = \cos\left(\frac{7}{2}\cdot\frac{\pi}{4}\right) = \cos\left(\frac{7\pi}{8}\right) = -\cos\left(\frac{\pi}{8}\right) = -\frac{\sqrt{\mathbf{2} + \sqrt{\mathbf{2}}}}{\mathbf{2}} \approx -0.92388$$

Let's verify:

$$T_4(x_k) = 8\left(\frac{\sqrt{2 \pm \sqrt{2}}}{2}\right)^4 - 8\left(\frac{\sqrt{2 \pm \sqrt{2}}}{2}\right)^2 + 1 = \frac{(2 \pm \sqrt{2})^2}{2} - 2(2 \pm \sqrt{2}) + 1 = \frac{6 \pm 4\sqrt{2}}{2} - 2(2 \pm \sqrt{2}) + 1 = 0$$

c) Transform these four nodes from the interval [-1, 1] to the interval [0, 6].

The midpoint is x_m =3, and the half-interval is also $h = 3$, therefore the Chebyshev points are:

$$x_{1,4} = 3\left(1 \pm \frac{\sqrt{2 + \sqrt{2}}}{2}\right),$$

$$x_{2,3} = 3\left(1 \pm \frac{\sqrt{2 - \sqrt{2}}}{2}\right).$$

Problem 3 (40pts) Chebyshev Interpolation

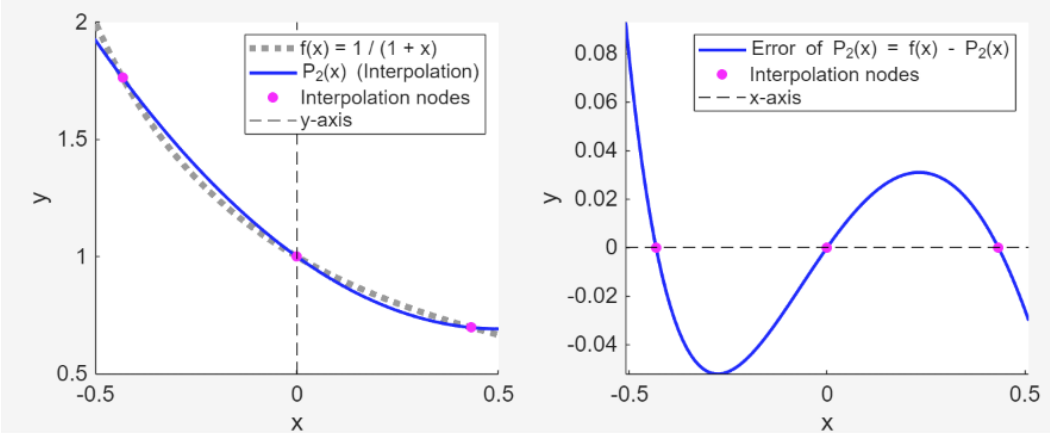
Now consider the same function $f(x) = \frac{1}{1+x}$ and interval, but use the three Chebyshev nodes: $\{x_k\} = \left\{-\frac{\sqrt{3}}{4},\ 0, \frac{\sqrt{3}}{4}\right\}$:

a) Find the quadratic interpolating polynomial $P_2(x)$ for this set of nodes.

x_k	$f[x_k]$	$f[x_{k-1}, x_k]$	$f[x_{k-2}, x_{k-1}, x_k]$
$-\frac{\sqrt{3}}{4}$	$\frac{1}{1-\frac{\sqrt{3}}{4}} = \frac{4}{4-\sqrt{3}}$		
0	1	$\frac{\left(1-\frac{4}{4-\sqrt{3}}\right)}{\sqrt{3}/4} = \frac{\left(-\frac{\sqrt{3}}{4-\sqrt{3}}\right)}{\sqrt{3}/4} = -\frac{4}{4-\sqrt{3}}$	
$\frac{\sqrt{3}}{4}$	$\frac{1}{1+\frac{\sqrt{3}}{4}} = \frac{4}{4+\sqrt{3}}$	$\frac{\left(\frac{4}{4+\sqrt{3}}-1\right)}{\sqrt{3}/4} = \frac{\left(-\frac{\sqrt{3}}{4+\sqrt{3}}\right)}{\sqrt{3}/4} = -\frac{4}{4+\sqrt{3}}$	$\frac{-\frac{4}{4+\sqrt{3}}+\frac{4}{4-\sqrt{3}}}{\sqrt{3}/2} = \frac{8}{\sqrt{3}}\left(-\frac{1}{4+\sqrt{3}}+\frac{1}{4-\sqrt{3}}\right) = \frac{16}{13}$

$$P_2(x) = f[x_1] + f[x_1,x_2](x-x_1) + f[x_1,x_2,x_3](x-x_1)(x-x_2) = \frac{4}{4-\sqrt{3}} - \frac{4}{4-\sqrt{3}}\left(x+\frac{\sqrt{3}}{4}\right) + \frac{16}{13}x\left(x+\frac{\sqrt{3}}{4}\right) = \boxed{1 - \frac{16}{13}x + \frac{16}{13}x^2}$$

b) Plot $f(x)$ and $P_2(x)$ together on $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ (use any computational tool). On a separate graph, plot the error $R_2(x) = f(x) - P_2(x)$, on the same interval.



c) Find an upper bound on the maximum absolute interpolation error, using the remainder formula from problem 1(b).

Hint: for Chebyshev nodes on [a, b], we've proven in class that $\max_{x\in[a,b]} |(x-x_1)\ldots(x-x_n)| = \frac{h^n}{2^{n-1}}$, where $h = \frac{b-a}{2}$.

$$\|R_2\|_\infty \le \underbrace{\left\|\frac{f'''(c)}{3!}\right\|_\infty}_{16} \|\psi\|_\infty = 16\,\frac{(1/2)^3}{4} = \mathbf{0.5} \qquad h^3/4$$

d) Compare the error bounds obtained in Problems 1(b) and 3(c). By what factor does the bound improve when equidistant nodes are replaced by Chebyshev nodes?

The error bound improved by about **35%**.

NOTE: The actual error didn't change much compared to Problem 1, since the interpolant is not optimal (not equioscillatory): Chebyshev nodes only minimize the nodal function. However, using more Chebyshev nodes and/or a small nudge of the nodes would greatly reduce the error.