**Read each problem carefully. Show all work for each problem. Electronic devices and notes *not* permitted. List of common Taylor expansions is given on page 2.**

**1) (14pts)** Find **two** terms in the Taylor expansion of $\boldsymbol{f(x) = \frac{2x^3}{1-\cos(2x)}}$ about $x = 0$, using composition of Taylor expansions of elementary functions (see page 2). Then, use your answer to estimate $f(\mathbf{0.3})$.

**Hint:** keep **3** terms in the expansion of $\cos(2x)$, and use the geometric series to obtain the reciprocal.

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**2) (12pts)** Does either of the functions below suffer loss of **all** significant digits when computed in double precision, assuming $|x| < 1$? If this does happen, estimate the corresponding range of $|\boldsymbol{x}|$ values:

$$(a)\ f(x) = \frac{\exp(4x^2) - 1}{x}, \qquad (b)\ g(x) = \frac{x^2}{x - \sin(2x^2)}.$$

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**3) (10pts)** Convert $x = \mathbf{30.625}$ from decimal to **binary** number system. Then, express this number in normalized binary scientific notation (a significand multiplied by an integer power of 2).

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**4) (14pts)** Consider the iteration $x_{n+1} = g(x_n)$, where $\boldsymbol{g(x) = \frac{x^2+12}{8}}$

**a)** Find all fixed points of this iteration, and determine their local stability.

**b)** For any stable fixed point *r*, estimate the number of iterations $\boldsymbol{n}$ needed to reduce the distance to the fixed point $r$ from $\boldsymbol{\epsilon_0 = \frac{1}{16}}$ to machine precision, $\boldsymbol{\epsilon_n \le 2^{-52}}$ (hint: no logarithms are needed).

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**5) (10pts)** Consider the iteration $x_{n+1} = g(x_n)$, where $g(x)$ is twice continuously differentiable near a fixed point r. Define $\epsilon_n \equiv x_n - r$.

**a)** Assuming $g'(r) \neq 0$, find the relationship between $\epsilon_{n+1}$ and $\epsilon_n$ using Taylor expansion of $g(x_n)$ at $r$.

**b)** Explain the local convergence or divergence criterion, assuming $|g'(r)| \neq 1$.

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**6) (16pts)** Consider the problem of numerically solving the equation $\boldsymbol{3x - x^2 = 1}$

**a)** Write down the Newton's iteration for solving this equation numerically.

**b)** Starting with $\boldsymbol{x_0 = 0}$, compute two iterates $x_1$ and $x_2$. You may leave your answer as fractions.

**c)** Using $\boldsymbol{r \approx 0.38}$ for the actual root, obtain a *rough* estimate for $|\epsilon_1| = |x_1 - r|$. Then, obtain a *rough* estimate of $|\epsilon_2|$ using the relationship $|\epsilon_2| \approx M\epsilon_1^2$, where $M = \left|\frac{f''(r)}{2f'(r)}\right|$.

**Note**: do ***not*** compute $|\epsilon_2|$ directly from $x_2$. Here "rough estimate" means 1 significant decimal digit.

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***Two more problems on the reverse side of this sheet***

**7) (16pts)** Consider the model

$$\boldsymbol{y_{model}(t;p,q) = \frac{p}{t} + q}$$

where $p$ and $q$ are unknown parameters.

**a)** Derive the linear system satisfied by the parameters $p$ and $q$ minimizing the sum of squared residuals with respect to the data set $\{(t_k, y_k)\}$ (use either calculus or the geometric approach).

**b)** Solve this system for the data set $\{(t_k, y_k)\} = \left\{\left(\frac{1}{3},\ 4\right), \left(\frac{1}{2},\ 3\right), (1,\ 1)\right\}$.

**c)** Compute all residuals, and plot the model and the data points.

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**8) (8pts)** Use the Lagrange polynomial basis to find an interpolating polynomial over the following 3 points:

$$\{(-1, 6), (1, 0), (2, 3)\}$$

You do **not** need to expand or simplify this polynomial.

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**List of Taylor expansions of common elementary functions:**

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + O\left(x^7\right) \qquad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O\left(x^6\right) \qquad \exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + O\left(x^4\right)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O\left(x^5\right) \quad \Rightarrow \quad \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + O\left(x^5\right)$$

$$(1 \pm x)^p = 1 \pm px + \frac{p(p-1)}{2}x^2 + O\left(x^3\right) \Rightarrow \begin{cases} p = \frac{1}{2}: & \sqrt{1 \pm x} = 1 \pm \frac{x}{2} - \frac{x^2}{8} + O\left(x^3\right) \\ p = -\frac{1}{2}: & \frac{1}{\sqrt{1 \pm x}} = 1 \mp \frac{x}{2} + \frac{3x^2}{8} + O\left(x^3\right) \\ p = -1: & \frac{1}{1 \pm x} = 1 \mp x + x^2 \mp x^3 + O\left(x^4\right) \end{cases}$$