1) (14pts) Find **two** terms in the Taylor expansion of $\boldsymbol{f(x) = \frac{2x^3}{1-\cos(2x)}}$ about $x = 0$, using composition of Taylor expansions of elementary functions (see page 2). Then, use your answer to estimate $f(\mathbf{0.3})$.

First, use expansion $\cos(u) \approx 1 - \frac{u^2}{2} + \frac{u^4}{4!}$ where $u = 2x$:

$$\cos(2x) \approx 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{4!} = 1 - 2x^2 + \frac{2}{3}x^4$$

Thus, the denominator equals

$$1 - \cos(2x) \approx 1 - \left(1 - 2x^2 + \frac{2}{3}x^4\right) = 2x^2 - \frac{2}{3}x^4$$

Now plug into $f(x)$:

$$f(x) = \frac{2x^3}{1-\cos(2x)} \approx \frac{2x^3}{2x^2 - \frac{2}{3}x^4} = \frac{x}{1 - \frac{x^2}{3}}$$

Finally, use the geometric series, $\frac{1}{1-u} \approx 1 + u$, where $u = \frac{x^2}{3}$:

$$f(x) \approx \frac{x}{1-\frac{x^2}{3}} = x\left(1 + \frac{x^2}{3}\right) = \boxed{x + \frac{x^3}{3}}$$

Now, plug in $x = 0.3$: $f(x) \approx 0.3 + \frac{(0.3)^3}{3} = 0.3 + 0.009 = \boxed{0.309}$ note: exact value is 0.309164... :)

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2) (12pts) Does either of the functions below suffer loss of **all** significant digits when computed in double precision, assuming $|x| < 1$? If this does happen, estimate the corresponding range of $|\boldsymbol{x}|$ values:

$$(a)\ f(x) = \frac{\exp(4x^2) - 1}{x}, \qquad (b)\ g(x) = \frac{x^2}{x - \sin(2x^2)}.$$

$$(a)\ f(x) = \frac{\exp(4x^2) - 1}{x} \approx \frac{(1 + 4x^2) - 1}{x} \to 0$$

Will be rounded off to **zero** (no correct significant digits) when $4x^2 \le 10^{-16} \Rightarrow \boxed{|x| \le \frac{10^{-8}}{2}}$

(*b*) No loss of all digits for $|x| \ll 1$: No work required (neither zero or ∞ is expected), but let's analyze this:

Two roundoff expected at different $|x|$ levels, both of which are extremely accurate and pose no issues:

$$g(x) = \frac{x^2}{x - \sin(2x^2)} \approx \frac{x^2}{x - \left(2x^2 - \frac{(2x^2)^3}{3!}\right)} \to \frac{x^2}{x - 2x^2} \to \frac{x^2}{x} = x \quad \text{(both reductions highly accurate!)}$$

NOTE: the only loss of accuracy is due to cancellation near $x^* = \sin(2x^{*2}) \approx 0.5$: **+3pts extra credit** if you noticed this! (since you were only expected to analyze the case of $|x|$ **much** smaller than 1)

3) (10pts) Convert $x = \mathbf{30.625}$ from decimal to **binary** number system. Then, express this number in normalized binary scientific notation (a significand multiplied by an integer power of 2).

$$30 = 16 + 8 + 4 + 2 = 11110_2$$

$$0.625 = 0.5 + 0.125 = 1/2 + 1/8 = 0.101_2$$

$$\mathbf{30.625_{10} = 11110.101_2 = 1.1110101_2 \times 2^4.}$$

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4) (14pts) Consider the iteration $x_{n+1} = g(x_n)$, where $\boldsymbol{g(x) = \frac{x^2+12}{8}}$

a) Find all fixed points of this iteration, and determine their local stability.

$$\text{(a)}\quad r = \frac{r^2+12}{8} \Rightarrow r^2 - 8r + 12 = 0 \Rightarrow (r-2)(r-6) = 0 \Rightarrow \begin{cases} r_1 = \mathbf{2}: \ |g'(r_1)| = \dfrac{2r_1}{8} = \dfrac{r_1}{4} = \mathbf{\dfrac{1}{2}} < 1: \ \text{Stable} \\ r_2 = \mathbf{6}: \ |g'(r_2)| = \dfrac{2r_2}{8} = \dfrac{r_2}{4} = \mathbf{\dfrac{3}{2}} > 1: \ \text{Unstable} \end{cases}$$

b) For any stable fixed point *r*, estimate the number of iterations $\boldsymbol{n}$ needed to reduce the distance to the fixed point r from $\boldsymbol{\epsilon_0 = \frac{1}{16}}$ to machine precision, $\boldsymbol{\epsilon_n \le 2^{-52}}$ (hint: no logarithms are needed).

$$C \equiv \left|g'(r_1)\right| = \frac{1}{2} \Rightarrow \varepsilon_{n+1} \approx C\,\varepsilon_n \quad \Rightarrow \quad \varepsilon_n \approx C^n \varepsilon_0 = \frac{1}{2^n}\cdot\frac{1}{16} = \frac{1}{2^{n+4}} \le \frac{1}{2^{52}} \Rightarrow n+4 \ge 52 \Rightarrow \boxed{n \ge 48}$$

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5) (10pts) Consider the iteration $x_{n+1} = g(x_n)$, where $g(x)$ is twice continuously differentiable near a fixed point r. Define $\epsilon_n \equiv x_n - r$.

a) Assuming $g'(r) \neq 0$, find the relationship between ϵ_{n+1} and ϵ_n using Taylor expansion of $g(x_n)$ at r.

b) Explain the local convergence or divergence criterion, assuming $|g'(r)| \neq 1$.

Plugging definition of error, $\boldsymbol{x_n = r + \epsilon_n}$, into our iteration $\boldsymbol{x_{n+1} = g(x_n)}$, we obtain:

$$\boldsymbol{x_{n+1} = r + \epsilon_{n+1} = g(r + \epsilon_n).}$$

Now, expand around *r* using the Taylor theorem, to lowest order:

$$\boldsymbol{r + \epsilon_{n+1} = g(r + \epsilon_n) = g(r) + g'(\alpha)\epsilon_n} \quad \text{where } \alpha \in (r, x_n)$$

Since by definition $\boldsymbol{r = g(r)}$, we can cancel ***r*** on both sides of this equation, leading to a simple result:

$$\boldsymbol{\epsilon_{n+1} = g'(\alpha)\epsilon_n}$$

Sufficiently close to the root $\alpha \approx r$, and we obtain an approximation: $\boldsymbol{\epsilon_{n+1} \approx g'(r)\epsilon_n}$

Therefore, the error contracts (or expands) by a factor $C \approx |g'(r)|$ at each iteration step.

- If $C < 1$, the error is contracting with each iteration, so the fixed point is stable
- If $C > 1$, the error is expanding with each iteration, so the fixed point is unstable

6) (16pts) Consider the problem of numerically solving the equation $\mathbf{3x - x^2 = 1}$

a) Write down the Newton's iteration for solving this equation numerically.

b) Starting with $\boldsymbol{x_0 = 0}$, compute two iterates x_1 and x_2. You may leave your answer as fractions.

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{3x - x^2 - 1}{3 - 2x} \quad \text{we could simplify this (optional):} \quad g(x) = \frac{1 - x^2}{3 - 2x}$$

$$x_0 = 0 \;\Rightarrow\; x_1 = 0 - \frac{0-1}{3-0} = \boxed{\frac{1}{3}} \;\Rightarrow\; x_2 = \frac{1}{3} - \frac{1 - \frac{1}{9} - 1}{3 - \frac{2}{3}} = \frac{1}{3} + \frac{1}{27 - 6} = \frac{1}{3} + \frac{1}{21} = \boxed{\frac{8}{21}} \approx 0.38095234$$

c) Using $\boldsymbol{r \approx 0.38}$ for the actual root, obtain a *rough* estimate for $|\epsilon_1| = |x_1 - r|$. Then, obtain a *rough* estimate of $|\epsilon_2|$ using the relationship $|\epsilon_2| \approx M\epsilon_1^2$, where $M = \left|\frac{f''(r)}{2f'(r)}\right|$.

$$|\varepsilon_1| = |x_1 - r| \approx |0.333\ldots - 0.38| \approx 0.05$$

$$M = \left|\frac{f''(r)}{2f'(r)}\right| = \left|\frac{-2}{2(3-2r)}\right| = \frac{1}{|3-2r|} \approx \frac{1}{3 - 2\cdot 0.38} = \frac{1}{2.24} \;\Rightarrow\; \varepsilon_2 \approx M\varepsilon_1^2 = \frac{0.0025}{2.24} \approx \boxed{0.001}$$ actual error is 0.0010134 :)

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7) (16pts) Consider the model $\boldsymbol{y_{model}(t; p, q) = \frac{p}{t} + q,}$ where p and q are unknown parameters.

a) Derive the linear system satisfied by the parameters p and q minimizing the sum of squared residuals with respect to the data set $\{(t_k, y_k)\}$ (use either calculus or the geometric approach).

b) Solve this system for the data set $\{(t_k, y_k)\} = \left\{\left(\frac{1}{3},\ 4\right), \left(\frac{1}{2},\ 3\right), (1,\ 1)\right\}$.

c) Compute all residuals, and plot the model and the data points.

The model basis functions are $\phi_1(t) = 1/t,\ \phi_2(t) = 1$, therefore our system is (after usual derivation):

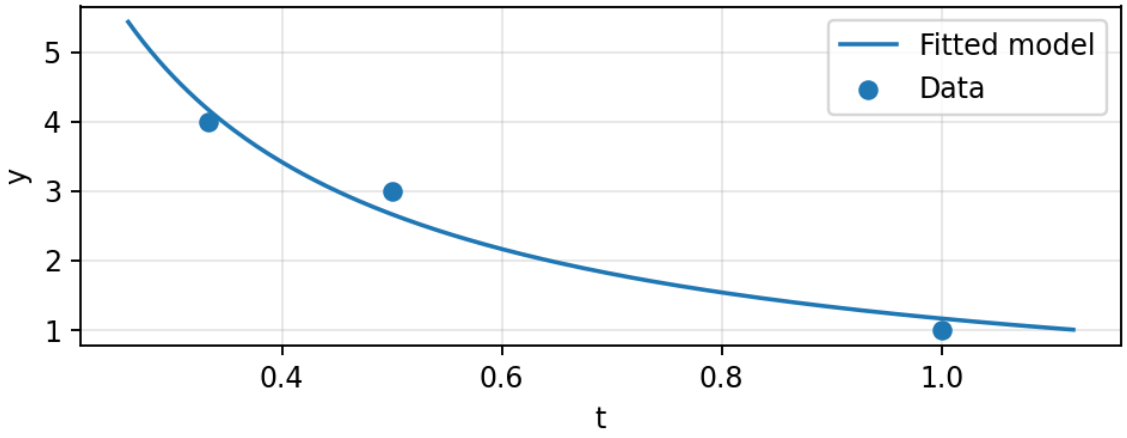
$$\begin{pmatrix} \sum_{k=1}^{N} \phi_1(t_k)^2 & \sum_{k=1}^{N} \phi_1(t_k)\phi_2(t_k) \\ \sum_{k=1}^{N} \phi_1(t_k)\phi_2(t_k) & \sum_{k=1}^{N} \phi_2(t_k)^2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{N} \frac{1}{t_k^2} & \sum_{k=1}^{N} \frac{1}{t_k} \\ \sum_{k=1}^{N} \frac{1}{t_k} & \sum_{k=1}^{N} 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{N} \frac{y_k}{t_k} \\ \sum_{k=1}^{N} y_k \end{pmatrix}$$ Then, plug in data points

Now plug in data points $\left\{\left(\frac{1}{3},\ 4\right), \left(\frac{1}{2},\ 3\right), (1,\ 1)\right\}$: $\left(\begin{array}{cc|c} 9+4+1 & 3+2+1 & 12+6+1 \\ 3+2+1 & 3 & 4+3+1 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 14 & 6 & 19 \\ 6 & 3 & 8 \end{array}\right)$

Only 2 simple row operations needed:

$$R_1 - 2R_2 \to R_1 \Rightarrow \left(\begin{array}{cc|c} 2 & 0 & 3 \\ 6 & 3 & 8 \end{array}\right) \Rightarrow R_2 - 3R_1 \to R_2 \Rightarrow \left(\begin{array}{cc|c} 2 & 0 & 3 \\ 0 & 3 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & -1/3 \end{array}\right) \Rightarrow \boxed{y_{\text{mod}} = \frac{3}{2t} - \frac{1}{3}}$$

Residuals sum to 0: $r_1 = \left(\frac{9}{2} - \frac{1}{3}\right) - 4 = \frac{25}{6} - 4 = \frac{1}{6};\quad r_2 = \left(3 - \frac{1}{3}\right) - 3 = -\frac{1}{3};\quad r_3 = \left(\frac{3}{2} - \frac{1}{3}\right) - 1 = \frac{7}{6} - 1 = \frac{1}{6}$



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8) (8pts) Use the Lagrange polynomial basis to find an interpolating polynomial over the following 3 points:

$$\{(-\mathbf{1},\mathbf{6}),(\mathbf{1},\mathbf{0}),(\mathbf{2},\mathbf{3})\}$$

You do **not** need to expand or simplify this polynomial.

$$f(x)\approx P(x)=6\cdot L_1(x)+0\cdot L_2(x)+3\cdot L_3(x)=\ 6\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}+3\frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

$$=6\frac{(x-1)(x-2)}{(-1-1)(-1-2)}+3\frac{(x+1)(x-1)}{(2+1)(2-1)}\ =\ \boxed{(x-1)(x-2)+(x+1)(x-1)}=(x-1)(2x-1)=\ 2x^2-3x+1.$$