Math 340 • Fall 2026 • Midterm Exam Practice Problems • Prof. Matveev

1) Find the 2nd degree Taylor polynomial for $\boldsymbol{f(x) = x^{-3}}$ near $x = 1$. Then, use your polynomial to approximate $f(1.1)$, and find the upper bound on the error of this approximation using the Taylor remainder formula.

2) Consider the function $\boldsymbol{f(x) = \dfrac{x^2}{x + \ln(\cos(x))}}$. Find the first two non-zero terms in the Taylor expansion of $f(x)$ near $x = 0$, using composition of Taylor series for the elementary functions appearing in this formula. Then, use your answer to estimate $f(0.2)$. **Hint:** the exact value (up to 4 digits) is $f(0.2) \approx 0.2224$.

3) Which of the functions below suffer catastrophic loss of accuracy (loss of all digits) when computed in double precision for small $|x|$? Determine approximately when this happens. What value will the computation return when this complete loss of significance happens?

(a) $f(x) = -4 + \sqrt{16 + x^4}$, (b) $g(x) = 4 + \sqrt{16 - x^2}$, (c) $h(x) = \dfrac{1}{6x - \sin(6x)}$

4) Convert $\boldsymbol{x = 134.3}$ from decimal to binary system, keeping 8 binary digits after the fractional point. Then, write this binary number in standard scientific notation (i.e. significant part times an integer power of 2).

5) Consider the root (zero) of function $f(x) = x^3 + x - 1$.

a) Find an obvious bracketing interval $[a, b]$ (with integer *a* and *b*) that contains the positive root of $f(x)$.

b) Perform two iterations of the bisection method, manually.

c) Write down the expression for the number of bisections of interval [*a*, *b*] needed to guarantee absolute error of 10^{-9} or smaller.

6) Consider the iteration $x_{n+1} = g(x_n)$, where $g(x) = \frac{1}{8}(x^2 + 7)$

a) Find all fixed points of this iteration, and determine their stability (**hint:** fixed points are integers)

b) For any **stable** fixed points, find the **number of steps** needed to achieve an absolute error (tolerance) of 2^{-52} (machine precision), assuming initial error of $\epsilon_0 \approx 1/4$. How does this compare with the number of iterations needed to achieve the same accuracy using the bisection method, assuming the same initial error?

7) Suppose the iteration $x_{n+1} = g(x_n)$ has a fixed point r where $g'(r) = 0$ and $g''(r) \neq 0$. Use Taylor series expansion of $g(x)$ about $x = r$ to determine how the error $\epsilon_n \equiv x_n - r$ changes from one step to the next.

8) Consider the problem of finding the root (zero) of function $f(x) = x^3 - 2$.

a) Set up the Newton's iteration formula for approximating the root of $f(x)$, and perform two iterations of the Newton's method to compute x_1 and x_2, given an initial guess $x_0 = 1$

b) Estimate the error ϵ_2 of approximant x_2 using $\epsilon_{n+1} \approx M\epsilon_n^2$ where $M = \frac{f''(r)}{2f'(r)}$, assuming that $\epsilon_0 \approx \frac{1}{4}$.

9) Find a function $f(x)$ whose root (solution to $f(r) = 0$) equals $r = 1/\sqrt{3}$. Your function should only involve multiplication, integer powers, and/or division (no fractional powers). Write down the Newton's iteration for this function, and perform two iterations to approximate $r = 1/\sqrt{3}$, starting with $x_0 = 1$.

10) Use the equation of the line through the points $(x_n, f(x_n))$ and $(x_{n-1}, f(x_{n-1}))$ to derive the formula for the **secant root-finding method**, which has the form $x_{n+1} = g(x_n, x_{n-1})$. Hint: intersection of this line with the horizontal axis is the next guess x_{n+1}.

11) The iteration $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)f(x_n)^2}{2[f'(x_n)]^3}$ is an improvement on the Newton's method, achieving higher order convergence. Perform one iteration using this formula with the function $f(x) = x^2 - 2$ and initial guess $x_0 = 1$, as a way to approximate $\sqrt{2}$.

12) Perform a least-squares fit of the model $\boldsymbol{y_{model}(t; p, q) = p \cdot 2^t + q \cdot t,}$ where p and q are unknown parameters, to the data set $\{(t_k, y_k)\} = \{(\mathbf{0}, \mathbf{1}), (\mathbf{1}, \mathbf{0}), (\mathbf{2}, \mathbf{5})\}$. Compute all residuals, and make sure to explain your solution method.

13) Consider fitting some data set $\{(t_k, y_k)\}$ to the model $\boldsymbol{y_{model}(t; C, p) = C\, t^p \cos(t),}$ where $\boldsymbol{C}$ and $\boldsymbol{p}$ are unknown parameters. Set up the linear algebra system necessary to determine the best values of these parameters, in the least-squares sense.

14) Use the Lagrange polynomial basis to find an interpolating polynomial over points $\{(-1,1), (0,0), (2,2), (4,6)\}$. You do **not** need to simplify.

15) The following MATLAB program computes an elementary function using its Taylor series expansion. Which function is it? (hint: follow the value of F through a couple iterations of the "while" loop)

```
function F = Taylor(x, eps)
    F = 1;       t = 1;
    x2 = x^2;   k = 1;
    while  abs(t) > eps
          t = – t * x2 / (k * (k + 1));
          F = F + t;
          k = k + 2;
    end
end
```