Section 3.4: Spline Interpolation.

Why use splines? (splines vs. polynomial interpolation)

Consider a trajectory of any physical body with an engine operated by a pilot, as a function of time. This trajectory obviously has a continuous first derivative (continuous velocity), and possibly continuous 2nd derivative (continuous acceleration, given by a continuous force), but there is no reason to expect a continuous 3rd derivative (known as "jerk"), since the force can change suddenly when the engine is started or suddenly stopped.

In fact, analyticity of trajectory (i.e. convergence of a Taylor series at all times) contradicts the pilot's ability to make decisions, since an analytic function is pre-determined by the values of its derivatives at any single time point!

In this example of a physical trajectory, polynomial approximation may still be very desirable. However, we can't increase approximation accuracy by using high-order polynomials: we may get spurious oscillations, like in the Runge's example.

Instead, we will break the time interval into sub-intervals, and use **splines: low-order *piece-wise* polynomial approximations that preserve continuity only for the lowest-order derivatives.**

Cubic splines are the most popular case, ensuring continuous 2nd derivative, but in many applications a **quadratic spline**, which only ensures continuous first derivative, is good enough.

Quadratic splines are also much easier than cubic splines, in that pieces can be added sequentially, one by one.

Quadratic Spline: an Example

Find a **quadratic spline** interpolation for the data points $\{(\boldsymbol{x_k}, \boldsymbol{y_k})\} = \{\boldsymbol{(0,0),(1,6),(2,2),(3,4)}\}$.

As will be shown below, we will need 1 extra condition, apart from enforcing continuity of the function and its derivative.

The general quadratic spline interpolating 4 points has the form

$$\begin{cases} s_1(x) = y_1 + a_1(x-x_1)^2 + b_1(x-x_1), & x \in [x_1, x_2] \\ s_2(x) = y_2 + a_2(x-x_2)^2 + b_2(x-x_2), & x \in [x_2, x_3] \\ s_3(x) = y_3 + a_3(x-x_3)^2 + b_3(x-x_3), & x \in [x_3, x_4] \end{cases}$$

Written in this form, the left-end point is satisfied automatically by each piece: note that $s_k(x_k) = y_k$.

Since there are **6** unknowns, we need **6** constraints:

- Continuity of the spline (**3** constraints), i.e. correct values at right endpoints: $\boldsymbol{s_k(x_{k+1}) = y_{k+1}}$
- Continuity of spline's derivative: interpolation at the right endpoints (**2** constraints): $\boldsymbol{s_k'(x_{k+1}) = s_{k+1}'(x_{k+1})}$
- **One** last constraint is needed: let's require zero derivative at the left endpoint, x_1: $s_1'(x_1) = 0$.

Start by plugging in the given spline points:

$$\begin{cases} s_1(x) = 0 + a_1x^2 + b_1x, & x \in [0, 1] \\ s_2(x) = 6 + a_2(x-1)^2 + b_2(x-1), & x \in [1, 2] \\ s_3(x) = 2 + a_3(x-2)^2 + b_3(x-2), & x \in [2, 3] \end{cases}$$

Spline piece #1: The extra condition (see above) $s_1'(0) = 0$ implies that $b_1 = 0$; therefore, we obtain

$$s_1(x) = a_1x^2$$

Let's fit the value of $y_2 = 6$ at $x_2 = 1$ (ensuring continuity of the spline at x_2): $s_1(1) = a_1 \cdot 1 = 6$

Thus, we know that

$$s_1(x) = 6x^2$$

Spline piece #2:

Now, proceed to the next piece: since $x_2 = 1,$ and $y_2 = 6$, we obtain:

$$s_2(x) = 6 + a_2(x-1)^2 + b_2(x-1)$$

Enforcing continuity of 1st derivative across $x_2 = 1,$ $s_2'(1) = s_1'(1),$ we obtain:

$$b_2 = 12$$

Finally, let's fit the value of $y_3 = 2$ at $x_3 = 2$ (ensuring the continuity of the spline at that point):

$$s_2(2) = 6 + a_2 + 12 = y_3 = 2$$

From this we obtain $a_2 = -16$

Thus, we know that

$$s_2(x) = 6 - 16(x-1)^2 + 12(x-1)$$

Spline piece #3:

Finally, let's determine the coefficients in the last piece of the spline:

$$s_3(x) = 2 + a_3(x-2)^2 + b_3(x-2)$$

Enforcing continuity of 1st derivative across the final internal node $x_3 = 2,$ $s_3'(2) = s_2'(2),$ we obtain:

$$b_3 = -32 + 12 = -20$$

Finally, let's fit the value of $y_4 = 4$ at right end-point $x_4 = 3$:

$$s_3(3) = 2 + a_3 - 20 = y_4 = 4$$

From this we obtain $a_3 = 22$

Thus, we know that

$$s_3(x) = 2 + 22(x-2)^2 - 20(x-2)$$

Let's combine all pieces together for our final answer, and plot:

$$\begin{cases} s_1(x) = 6x^2, & x \in [0,1] \\ s_2(x) = 6 - 16(x-1)^2 + 12(x-1), & x \in [1,2] \\ s_3(x) = 2 + 22(x-2)^2 - 20(x-2), & x \in [2,3] \end{cases}$$

