Review of Calculus

Textbook section 0.5

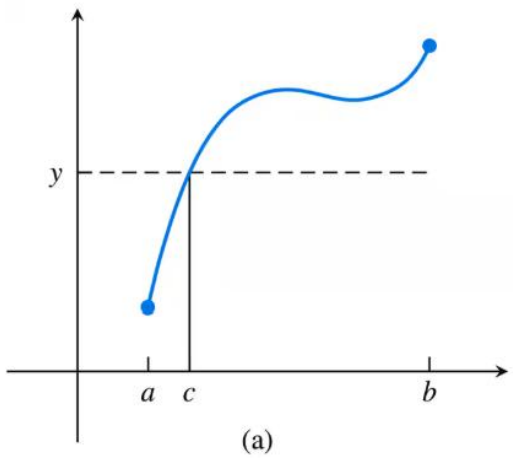
Let's review some simple but at the same time very important theorems from Calculus that will help understand better the course material. There are really only a couple of these results that we have to review.

THEOREM 1 (0.5): Intermediate Value Theorem

If $f(x)$ is continuous on $[a, b]$, then $f(x)$ achieves every value from $f(a)$ to $f(b)$.

Which means, in mathematical language:

If $f(x) \subset C_{[a,b]}$ then for every $y \in [f(a), f(b)]$ there exists $c \in [a, b]$ such that $f(c) = y$:



(a)

Put this in another, simpler way: a continuous function can have no gaps in its range.

Example 0.7

Show that $f(x) = x^2 - 3$ on the interval $[1, 3]$ must take on the values 0 and 1.

Because $f(1) = -2$ and $f(3) = 6$, all values between -2 and 6, including 0 and 1, must be taken on by f. For example, setting $c = \sqrt{3}$, note that $f(c) = f(\sqrt{3}) = 0$, and secondly, $f(2) = 1$.

Corollary for root-finding (which is why we review it!)

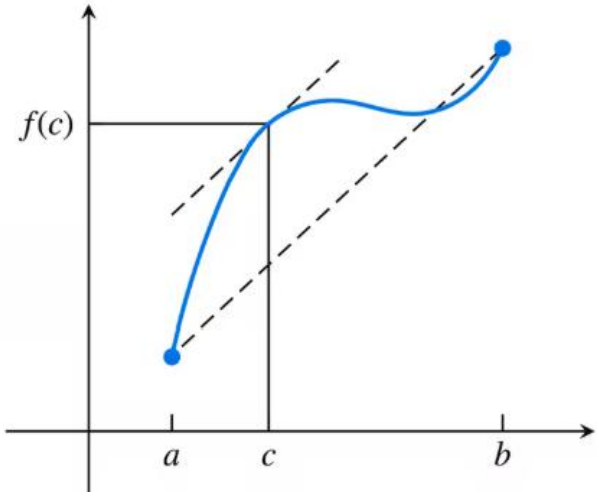
If $f(x)$ is continuous on $[a, b]$, and $f(a)f(b) < 0$ (i.e. $f(x)$ has opposite signs at the endpoints), then $f(x)$ has a root (a zero) within (a, b).

THEOREM 2 (0.6): Mean-Value Theorem

If $f(x)$ is continuous on [a, b] and has a continuous derivative on (a, b), then $f(x)$ achieves its average slope somewhere between a to b.

In mathematical language:

If $f(x) \subset C_{[a,b]}$ and $f(x) \subset C^1_{(a,b)}$ then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$:



Example 0.8

Apply the Mean Value Theorem to $f(x) = x^2 - 3$ on the interval $[1, 3]$.

The content of the theorem is that because $f(1) = -2$ and $f(3) = 6$, there must exist a number c in the interval (1, 3) satisfying $f'(c) = (6 - (-2))/(3 - 1) = 4$. It is easy to find such a c. Since $f'(x) = 2x$, the correct $c = 2$.

MVT version with f(a) = f(b) is called the *Rolle's Theorem*:

If $f(x) \subset C_{[a,b]}$, $f(x) \subset C^1_{(a,b)}$ and $f(a) = f(b)$ then there is a $c \in (a, b)$ where $f'(c) = 0$.

Easy to understand this geometrically: if a smooth function has the same values at the endpoints of an interval, it has at least one local minimum or one local maximum on that interval, where the derivative is zero.

Connection of MVT to Taylor Series

Despite its simplicity (it is almost obvious!), MVT is a building block for many important results, including one of the most important results in the entire subject of Calculus: the Taylor Series.

Consider again the MVT: if $f(x) \subset C^1_{(a,b)}$ then exists $c \in (a,b)$ so that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Solving for f(b), we obtain:

$$f(b) = f(a) + f'(c)(b-a)$$

This in fact is the lowest-order (k = 0) Taylor series, with the last term representing the "remainder".

Applying the MVT to higher derivatives of a function that is differentiable k+1 times (i.e. $(f(x) \subset C^{k+1}_{(a,b)})$, it is possible to obtain the Taylor series of order k:

Theorem 3 (0.8): Taylor Series

If $f(x) \subset C^{k+1}_{(a,b)}$ (function has a continuous derivative of order k+1), then there exists $c \in (a,b)$ so that:

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2}(b-a)^2 + \frac{f'''(a)}{3!}(b-a)^3 + \cdots + \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(k+1)}(c)}{(k+1)!}(b-a)^{k+1}$$

The Taylor series will be explored more in a separate lecture note; on the next page is the last corollary of the MVT that we need to review.

Theorem 4 (0.9): Average Value Theorem

If one applies the Mean Value Theorem to a derivative of a function, one obtains the Average Value Theorem, also known as "MVT for Integrals":

Consider a function $F(x)$ such that $F'(x) = f(x)$. Then, using the Fundamental Theorem of Calculus and the MVT statement that $f'(c) = \frac{f(b)-f(a)}{b-a}$ we obtain:

If $f'(x) = F(x)$ is continuous on $[a, b]$, then there exists $c \in (a, b)$ such that:

$$F(c) = \frac{f(b)-f(a)}{b-a} = \frac{1}{b-a}\int_a^b f'(x)dx = \frac{1}{b-a}\int_a^b F(x)dx$$

The value on the right hand side represents the average value of the function F(x) over the given interval (the integral divided by the length of the integration domain).

Thus, the theorem states that a function continuous on a given interval achieves its average value somewhere on that same interval.

Our textbook also mentions a somewhat more general version of the Average Value Theorem, but up to a change of variables (substitution) $dy = g(x)dx$, it actually represents the same result:

Theorem 5 (0.9'): Average Value Theorem (general form)

(Mean Value Theorem for Integrals) Let f be a continuous function on the interval $[a, b]$, and let g be an integrable function that does not change sign on $[a,\ b]$. Then there exists a number c between a and b such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

That's about it! See another handout for a more thorough review of the Taylor Series.