Math 340 • Fall 2026 • NJIT • Victor Matveev

Review of Calculus: Taylor Series

Textbook section 0.5

Taylor series is one of the most important results arising from the invention of Calculus, and is also one of the most practical ones: our entire course is one long application of the Taylor Series.

Taylor's theorem states that any sufficiently smooth function (regardless of how "complicated" it is, and even if it has no closed-form expression) can be approximated by a simple polynomial!

Theorem 3 (0.8): Taylor Series

If $f(x)$ is continuous on $[x_0, x]$ and is differentiable $k+1$ times between x_0 and x, then

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_k(x)$$

The last term is the remainder of this degree-*k* Taylor polynomial: $R_k(x) = \frac{f^{(k+1)}(c)}{(k+1)!}(x-x_0)^{k+1}$

The constant c in this remainder formula lies somewhere within the interval between x_0 and x.

Note: the theorem gives no prescription on how to find argument value *c* in the remainder, but even without its knowledge it allows us to estimate (to find an upper bound for) the remainder (see example on next page).

Proof: keep integrating by parts the fundamental theorem of calculus $\int_{x_0}^{x} f'(t)\,dt = f(x) - f(x_0)$, choosing $u = f(t), v = t - x, du = f'(t)dt,\ dv = dt.$

NOTE: the following Taylor series formulas are completely equivalent to above:

Defining $\boldsymbol{x = x_0 + h}$, we view the Taylor series as an expansion in small displacement h around point x_0:

$$f(x_0+h) = f(x_0) + f'(x_0)\,h + \frac{f''(x_0)}{2}h^2 + \frac{f'''(x_0)}{3!}h^3 + \cdots + \frac{f^{(k)}(x_0)}{k!}h^k + \frac{f^{(k+1)}(c)}{(k+1)!}h^{k+1}$$

Denoting $\boldsymbol{x_0 = a},\ \boldsymbol{h = x}$ better illustrates the fact that the truncated Taylor series is just a simple polynomial:

$$f(a+x) = f(a) + f'(a)\,x + \frac{f''(a)}{2}x^2 + \frac{f'''(a)}{3!}x^3 + \cdots + \frac{f^{(k)}(a)}{k!}x^k + \frac{f^{(k+1)}(c_x)}{(k+1)!}x^{k+1}$$

Further, we can always shift the x-axis so that $\boldsymbol{x_0 = a = 0}$, which gives the so-called Maclaurin series (which is not really deserving any special name since it is completely equivalent to above, up to a variable re-definition):

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \frac{f^{(k+1)}(c_x)}{(k+1)!}x^{k+1}$$

The unknown constant c_x in the remainder lies somewhere between $a = 0$ and x, so its value depends on x.

The following example shows how to use the remainder to bound the error of the Taylor polynomial approximation:

Example 1.1 (example 0.9 in the textbook):

- Find the degree 4 Taylor polynomial $\boldsymbol{P_4(x)}$ for $\boldsymbol{f(x) = \sin(x)}$ centered at the point $x_0 = 0$.
- Estimate the **maximum error** when using $P_4(x)$ to estimate $\sin(x)$ for $|x| \leq 0.001$.

Solution:

Even derivatives of $f(x)$ at $x = 0$ are $f^{(2m)}(0) = (-1)^m \sin(0) = 0$, while odd derivatives are $f^{(2m+1)}(0) = (-1)^m \cos(0) = (-1)^m$, therefore $\boldsymbol{P_4(x) = x - \frac{x^3}{3!} = x - \frac{x^3}{6}}$ with the remainder $\boldsymbol{R_4(x) = \frac{x^5}{5!} f^5(c) = \frac{x^5}{120} \cos(c)}$, which in absolute value cannot be larger than $|x|^5/120$. For $|x| < 0.0001$, the remainder is at most $10^{-20}/120$.

Forming Taylor polynomials by composition of known series

Practically, in most cases it is easier to find the Taylor Series of a given function by combining known series of the most widely used smooth functions listed below, instead of differentiation. You should remember the following by heart, since they appear in so many applications (here *a*=0 in all cases):

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7) \qquad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^6) \qquad \exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + O(x^4)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^5) \quad \Rightarrow \quad \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + O(x^5)$$

$$(1 \pm x)^p = 1 \pm px + \frac{p(p-1)}{2} x^2 + O(x^3) \Rightarrow \begin{cases} p = \frac{1}{2}: & \sqrt{1 \pm x} = 1 \pm \frac{x}{2} - \frac{x^2}{8} + O(x^3) \\ p = -\frac{1}{2}: & \frac{1}{\sqrt{1 \pm x}} = 1 \mp \frac{x}{2} + \frac{3x^2}{8} + O(x^3) \\ p = -1: & \frac{1}{1 \pm x} = 1 \mp x + x^2 \mp x^3 + O(x^4) \end{cases}$$

As an exercise, prove the above formulas by differentiating each function a couple of times.

Important note: A given Taylor series does not necessary converge for *all* values of *x*. While the first three series above do converge everywhere, the rest only converge for $|x| < 1$ (due to the singularity at $x = \pm 1$).

Below are a couple examples showing how to obtain the first few terms in the power expansion of a nested function by composing known Taylor series on the previous page:

$$\text{Example 1.2: } f(x) = \exp(\sin x) \approx 1 + \sin x + \frac{\sin^2 x}{2!} + O(x^3)$$

$$= 1 + \left(x - \frac{x^3}{3!} + O(x^5) \right) + \frac{1}{2} \left(x - \frac{x^3}{3!} + O(x^5) \right)^2 + O(x^3) = 1 + x + \frac{x^2}{2} + O(x^3)$$

Here we combined the Taylor Series for sin(x) = x - x^3 /6 + … with the exponential series exp(u) = 1 + u + u^2/2 +… The "big-O" notation O(x^3) denotes terms of order x^3.

Another example:

$$\text{Example 1.3:}\quad f(x)=\sec(x)\equiv\frac{1}{\cos x}=\frac{1}{1-\underbrace{\left(\frac{x^2}{2}-\frac{x^4}{4!}+O\left(x^6\right)\right)}_{u}}=1+u+u^2+O\left(u^3\right)$$

$$=1+\frac{x^2}{2}-\frac{x^4}{4!}+\left(\frac{x^2}{2}+O\left(x^4\right)\right)^2+O\left(x^6\right)$$

$$=1+\frac{x^2}{2}+x^4\left(\frac{1}{4}-\frac{1}{4!}\right)+O\left(x^6\right)=1+\frac{x^2}{2}+\frac{5x^4}{24}+O\left(x^3\right)$$

Here we combined the series for cos(x) = 1 – x^2 /2 + x^4/4! + … with the geometric series 1/(1 – u) = 1 + u + u^2+… Of course, you can verify by differentiation that these expansions are valid. Let's do one more example:

$$\text{Example 1.4:}\quad f(x)=\frac{1}{\sqrt{1+\sin x}}\equiv\left[1+\sin x\right]^{-1/2}=1-\frac{1}{2}\sin x+\frac{3}{8}\sin^2 x+O\left(x^3\right)$$

$$=1-\frac{1}{2}\left(x-\frac{x^3}{3!}+O\left(x^5\right)\right)+\frac{3}{8}\left(x-\frac{x^3}{3!}+O\left(x^5\right)\right)^2+O\left(x^3\right)=1-\frac{x}{2}+\frac{3x^2}{8}+O\left(x^3\right)$$

Here we combined the Taylor Series for sin(x) and the series $(1+u)^{-1/2}$ = 1 – u/2 + $3u^2$/8 - …

Example 1.5

There *are* however cases when direct differentiation may be more convenient than nesting several Taylor series. Consider for instance the lowest-order Taylor polynomial for $f(x) = 1/(1+x^2)$ near a = 1. We could in fact use the geometric series, but would have to do a variable shift first (since the expansion is not around a=0). This is why it may be easier to just use the Taylor formula, at least for the first term (i.e. for the linearization):

$$f(x)=\frac{1}{1+x^2}\Rightarrow f'(x)=-\frac{2x}{\left(1+x^2\right)^2}\Rightarrow f'(1)=-\frac{2}{4}=-\frac{1}{2},\quad f(1)=\frac{1}{2}\Rightarrow f(x)=\frac{1}{2}-\frac{x-1}{2}+O\left((x-1)^2\right)$$

Application 1

One of important applications of the Taylor Series is in approximating the function value by computing its Taylor polynomial, as was already shown above in example 1.1. Let's do a couple more examples of this kind:

Example 1.6:

For instance, considering the 3rd example on the preceding page (example 1.4), we can easily estimate $f(0.2)$:

$$f(0.2) = \frac{1}{\sqrt{1+\sin 0.2}} \approx 1 - \frac{0.2}{2} + \frac{3(0.2)^2}{8} = 0.9 + \frac{0.12}{8} = 0.915$$

A more exact result is $f(0.2) = 0.9133775\ldots$, so the absolute error is quite small, about 0.0016, and the relative error is only 0.18%. Note that we could have estimated the error *a priori* using the remainder formula. Let's do so in the next example.

Example 1.7:

Given the function $\boldsymbol{f(x) = \sqrt[3]{1+x}}$, do the following :

1) Construct the 2nd order Taylor polynomial for $f(x)$ around $x = 0$.

2) Use the polynomial to estimate $(0.3) = \sqrt[3]{1.3}$.

3) Find the maximal error (i.e. the error bound) for this estimate.

Solution:

1)
$$\begin{cases} f(x) = (1+x)^{1/3} \\ f'(x) = \frac{1}{3}(1+x)^{-2/3} \\ f''(x) = -\frac{2}{9}(1+x)^{-5/3} \\ f'''(x) = \frac{10}{27}(1+x)^{-8/3} \end{cases} \Rightarrow \begin{cases} f(0) = 1 \\ f'(0) = \frac{1}{3} \\ f''(0) = -\frac{2}{9} \\ f'''(c) = \frac{10}{27}(1+c)^{-8/3} \end{cases} \Rightarrow$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(c)}{3!}x^3$$
$$= \underbrace{1 + \frac{x}{3} - \frac{x^2}{9}}_{\text{Taylor polynomial } P_2(x)} + \underbrace{\frac{10}{6\cdot 27}\frac{x^3}{(1+c)^{8/3}}}_{\text{Remainder (error) } R_2(x)}$$

2) Therefore, we find that, approximately, $\sqrt[3]{1.3} \approx 1 + \frac{0.3}{3} - \frac{0.3^2}{9} = 1.1 - 0.01 = 1.09$

3) Finally, let's estimate the maximal error (this is called "bounding the error"). Since *c* lies somewhere between 0 and the function argument 0.3, and the denominator is an increasing function of *c*, plugging in the smallest values of $c = 0$ gives the largest error. In other words:

$$R_2(0.3) = \frac{10}{6\cdot 27}\frac{0.3^3}{(1+c)^{8/3}} \le \frac{10\cdot 0.3^3}{6\cdot 27} = \frac{0.01}{6} = 0.00\overline{16}.$$

Note that the error is positive, therefore the true function value should satisfy $1.09 \le f(0.3) \le 1.0917$

Our error estimate turns out to be quite accurate, since the exact value is $\sqrt[3]{1.3} = 1.091392883$: nice! ☺

See the plot of the function compared to its quadratic approximation: https://tinyurl.com/5x2h5knh

Application 2 (more conceptual and fun)

The following Taylor Series result you are all familiar with, even if you did not realize it has anything to do with the Taylor Series!

Namely, you learned in physics class that $\frac{mv^2}{2}$ represents the kinetic energy of a body of mass *m* moving with speed *v*. As it turns out, this is just an approximation: it's the quadratic Taylor series term in the expansion of more accurate relativistic energy in powers of speed. In fact, as the speed approaches the speed of light, the energy should approach infinity (not $mc^2/2$) , since such a speed is impossible to attain with a finite energy.

This fact is captured by the well-known increase of mass of a fast-moving body due to relativistic effects, as shown below, which we then expand using the Taylor series for $(1-x)^{-1/2}$ in our table on page 2:

$$E = mc^2 \equiv \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} = m_0 c^2 \left(1-\frac{v^2}{c^2}\right)^{-1/2} = m_0 c^2 \left[1+\frac{v^2}{2c^2}+\frac{3}{8}\frac{v^4}{c^4}+O\left(\frac{v^6}{c^6}\right)\right]$$

$$= \underbrace{m_0 c^2}_{\text{Rest energy}} + \underbrace{\frac{m_0 v^2}{2}\left(1+\frac{3}{8}\frac{v^2}{c^2}+O\left(\frac{v^4}{c^4}\right)\right)}_{\text{Kinetic energy}} = m_0 c^2 + \frac{m_0 v^2}{2} + \underbrace{\frac{3m_0 v^4}{16c^2}}_{\substack{\text{Relativistic}\\ \text{correction}}} + O\left(v^6\right)$$

where m_0 is called the "resting mass" of a body (mass at zero velocity), and we used the following Taylor series from our table:

$$(1-x)^{-1/2} = 1-\frac{x}{2}+\frac{3}{8}x^2+O\left(x^3\right)$$

where $x \equiv v^2/c^2$.